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WEIGHTS FOR CLASSICAL GROUPS

JIANBEI AN

ABSTRACT. This paper proves the Alperin's weight conjecture for the finite unitary groups when the characteristic r of modular representation is odd. Moreover, this paper proves the conjecture for finite odd dimensional special orthogonal groups and gives a combinatorial way to count the number of weights, block by block, for finite symplectic and even dimensional special orthogonal groups when r and the defining characteristic of the groups are odd.

INTRODUCTION

Let G be a finite group and r a prime. A *weight* of G is a pair (R, φ) of an r-subgroup R of G and an irreducible character φ of $N(R)$ such that φ is trivial on R and in an r-block of defect 0 of $N(R)/R$, where $N(R) = N_G(R)$ is the normalizer of R in G. A *radical* subgroup R of G is an r-subgroup of G such that $R = O_r(N(R))$, where $O_r(N(R))$ is the largest normal r-subgroup of $N(R)$. If (R, φ) is a weight of G, then R is necessarily a radical subgroup of G. A weight (R, φ) is a *B-weight* for an r-block B of G if φ is contained in an r-block b of $N(R)$ such that $B = b^G$, that is, B corresponds to b by the Brauer homomorphism. In his paper [2], Alperin introduced the concept of weight in the modular representation theory of finite groups and conjectured that the number of weights of G should equal the number of modular irreducible representations. Moreover, this equality should hold block by block. Here a weight (R, φ) is identified with its conjugates in G. Alperin and Fong in [3] have proved this conjecture for symmetric groups and for finite general linear groups when the characteristic r of modular representation is odd. The author in [4, 5] proved the conjecture for finite general linear and unitary groups when r is even. In this paper, we prove the conjecture for the finite unitary groups when r is odd. Moreover, we prove the conjecture for odd dimensional special orthogonal groups and give a combinatorial way to count the number of weights, block by block, for both finite symplectic and even dimensional special orthogonal groups when r and the defining characteristic p of groups are odd. We may suppose p is different from r since the result is known when p is r (see [2]).

In the first two sections, we describe the local structures of radical subgroups of a finite classical group, and in §3 we count the number of weights when the center of a radical subgroup is cyclic. The conjecture has been proved for unitary groups in (4D) and for odd dimensional special orthogonal groups in

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(4G) and its remarks. Finally, the numbers of weights for symplectic and even dimensional special orthogonal groups have been counted in (4F) and (4H) respectively.

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1. THE GROUPS OF SYMPLECTIC TYPE

Throughout this paper we shall follow the notation of [3, 5, 7], and [12]. In particular, r is an odd prime and E is an extraspecial r-group of order $r^{2\gamma+1}$ with center $Z(E) = \langle y \rangle$. Then $E = \langle x_1, x_2, \dots, x_{2\gamma-1}, x_{2\gamma} \rangle$ such that $[x_{2i-1}, x_{2i}] = x_{2i-1}^{-1} x_{2i}^{-1} x_{2i-1} x_{2i} = y$, $[x_{2i}, x_{2i+1}] = 1$, for $1 \leq i \leq \gamma$, $[x_i, x_j] = 1$ for $|i-j| \geq 2$, $x_i^r = 1$ for $i \neq 2$. Thus E has exponent r or r^2 according as $x_2^r = 1$ or y. An r-group R is of *symplectic type* if R is a central product of a nontrivial cyclic r-group Z and an extraspecial group E, where $Z(E)$ is identified with $\Omega_1(Z)$. If $R > E$, then R can be rewritten as the central product of Z and an extraspecial group of exponent r, so that we may suppose E has exponent r and $E = \Omega_1(R)$. Thus we may always suppose E is characteristic in R. Let $\operatorname{Aut} R$ be the automorphism group of R, $\operatorname{Inn} R$ the group of inner automorphisms, and $\operatorname{Aut}^0 R = \{\sigma \in \operatorname{Aut} R : [\sigma, Z] = 1\}$. Since every σ in $\operatorname{Aut}^0 R$ restricts to an element $\operatorname{Aut}^0 E$ and every σ in $\operatorname{Aut}^0 E$ extends to an element of $\operatorname{Aut}^0 R$, it follows that $\operatorname{Aut}^0 R = \operatorname{Aut}^0 E$. Denote

$$(1.1) \qquad K = \begin{cases} \operatorname{Sp}(2\gamma, r) & \text{if } E \text{ has exponent } r, \\ \operatorname{Sp}(2\gamma-2, r) \ltimes r^{2(\gamma-1)+1} & \text{if } E \text{ has exponent } r^2, \end{cases}$$

where $r^{2(\gamma-1)+1}$ denotes the extraspecial group of order $2(\gamma-1)+1$ and exponent r, and $\operatorname{Sp}(0, r) \ltimes r^1$ is interpreted as a group of order r. By [20, Theorem 1 or 15, p. 404] $\operatorname{Aut}^0 E = K \ltimes \operatorname{Inn} E$ (see also [3, p. 10]). In the following we shall consider the embeddings of R into classical groups and determine the local structures of these embeddings.

Let $\mathbb{F}_q$ be the field of q elements and $\eta = \pm 1$ a sign, where q is a power of prime p distinct from r. We first consider the embedding of E in the groups $G = \operatorname{GL}(n, \eta q)$. Here following [7], we denote $\operatorname{U}(n, q)$ by $\operatorname{GL}(n, -q)$. The proofs of the following two lemmas are similar to that of [5, (1D), (1E), and (1F)] and in the proofs such terms as orthogonal, orthonormal, and isometric will have meaning only in contexts involving $\operatorname{U}(n, q)$ and unitary spaces, but no meaning in contexts involving $\operatorname{GL}(n, q)$ and linear spaces.

(1A). *Let E be an extraspecial group of order $r^{2\gamma+1}$ and $G = \operatorname{GL}(r^\gamma, \eta q)$. If r divides $q - \eta$ (written $r | q - \eta$), then G contains a unique conjugacy class of subgroups isomorphic to E. Moreover, if $r | q - 1$, then $\mathbb{F}_q$ is a splitting field of E.*

Proof. Given $1 \leq i \leq \gamma$, let $E_i = \langle x_{2i-1}, x_{2i} \rangle$, and V_i a linear space of dimension r over $\mathbb{F}_q$ or a unitary space of dimension r over $\mathbb{F}_{q^2}$ according as $\eta = 1$ or -1. Then E_i acts faithfully, irreducibly, and isometrically on V_i. Namely, let w be an rth root of unity in $\mathbb{F}_{q^2}$ and $\{v_1^i, v_2^i, \dots, v_r^i\}$ an orthonormal basis of V_i. If E has exponent r, then define

$$(1.2) \qquad x_{2i-1} : v_j^i \mapsto w^j v_j^i, \qquad x_{2i} : v_j^i \mapsto v_{j+1}^i,$$

where $1 \leq j \leq r$. If E has exponent r^2, then define

$$(1.3) \qquad x_{2i-1}\colon v_j^i \mapsto w^j v_j^i, \qquad x_{2i}\colon v_j^i \mapsto \begin{cases} wv_1^1 & \text{if } i = 1 \text{ and } j = r, \\ v_{j+1}^i & \text{otherwise,} \end{cases}$$

where $1 \leq j \leq r$. Here subscripts on basis vectors are naturally read modulo r. In particular, $y\colon v_j^i \mapsto wv_j^i$ for all j.

Since E is the central product of the E_i's and the element y in $Z(E_i)$ is represented on V_i by the scalar matrix wI, E acts faithfully and irreducibly on $V = V_1 \otimes V_2 \otimes \cdots \otimes V_\gamma$. To see that the actions are by isometries, we first simplify notation and write

$$v_{j_1}^1 \otimes v_{j_2}^2 \otimes \cdots \otimes v_{j_\gamma}^\gamma = [j_1, j_2, \ldots, j_\gamma], \qquad 1 \leq j_i \leq r.$$

The r^γ elements $[j_1, j_2, \ldots, j_\gamma]$ form an orthonormal basis for V. So

$$(1.4) \qquad \begin{aligned} x_{2i-1}&\colon [j_1, j_2, \ldots, j_\gamma] \mapsto w^{j_i}[j_1, j_2, \ldots, j_\gamma], \\ x_{2i}&\colon [j_1, j_2, \ldots, j_\gamma] \mapsto [j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_\gamma], \end{aligned}$$

except when E has exponent r^2, in which case the actions of x_i for $i \neq 2$ are given by (1.4) and

$$(1.5) \qquad x_2\colon [j_1, j_2, \ldots, j_\gamma] \mapsto \begin{cases} [j_1 + 1, j_2, \ldots, j_\gamma] & \text{if } j_1 \neq r, \\ w[1, j_2, \ldots, j_\gamma] & \text{if } j_1 = r. \end{cases}$$

Since basic vectors are mapped onto orthonormal vectors by generating elements of E, E acts on V by isometries, so that G contains a copy of E.

Suppose $r | q - 1$. Replacing w by w^k for $1 \leq k < r$ in the proof above, we get $r - 1$ faithful and irreducible representations of E. By [14, 5.5.4] E has $r - 1$ nonlinear characters and all linear characters are realizable over $\mathbb{F}_q$ since $E/Z(E)$ is an elementary abelian r-group. Thus $\mathbb{F}_q$ is a splitting field of E.

To prove the uniqueness, it suffices to show that if E is embedded as a subgroup of G, then there exists an orthonormal basis of the underlying space V such that (1.4) or (1.5) holds according as E has exponent r or r^2. By Schur's lemma $y = w^k I$ for some integer $1 \leq k < r$. We may suppose $y = wI$ since $E = \langle x_1, x_2^k, x_3, x_4^k, \ldots, x_{2\gamma-1}, x_{2\gamma}^k \rangle$ and $[x_{2i-1}, x_{2i}^k] = y^k$.

Let $W_j = \{v \in V\colon x_1 v = w^j v\}$ for $1 \leq j \leq r$. Then V is the orthogonal sum of the W_j, so the W_j for $1 \leq j \leq r$ are nondegenerate subspaces of V and they are permuted by x_2 cyclically

$$x_2 W_1 = W_2, \qquad x_2^2 W_1 = W_3, \ldots, x_2^r W_1 = W_1,$$

since $x_1 x_2 = w x_2 x_1$. In particular, W_j for $1 \leq j \leq r$ have the same dimension.

If $\gamma = 1$ and $\{v_1\}$ is an orthonormal basis of W_1, then $\{v_1, x_2 v_1, \ldots, x_2^{r-1} v_1\}$ is an orthonormal basis of V and the actions of x_1 and x_2 on the basis are given by (1.2) or (1.3) according as E has exponent r or r^2. If $\gamma \geq 2$, then $L = \langle x_3, x_4, \ldots, x_{2\gamma} \rangle$ is an extraspecial group of order $r^{2\gamma-1}$ and exponent r acting faithfully on W_1. We may suppose by induction that $x_3, x_4, \ldots, x_{2\gamma}$ act on W_1 by (1.4) relative to the orthonormal basis $\{[j_2, j_3, \ldots, j_\gamma]\}$ of W_1, where $1 \leq j_i \leq r$. Thus $\{[j_1, j_2, \ldots, j_\gamma] = x_2^{j_1 - 1}[j_2, \ldots, j_\gamma]\colon 1 \leq j_i \leq r\}$ is an orthonormal basis of V and $x_1, x_2, \ldots, x_{2\gamma}$ act on the basis by (1.4) or (1.5). Thus any two embeddings of E in G are conjugate.

Remark. (1) Suppose $r|q-\eta$ and E is embedded in $G = \mathrm{GL}(n, \eta q)$ as a subgroup such that y is represented by a scalar multiple of the identity matrix. Then $n = mr^\gamma$ for some integer $m \geq 1$, and there exists an orthonormal basis $\{[j_1, j_2, \ldots, j_\gamma]_k\}$ of the underlying space V of G, where $1 \leq j_i \leq r$ and $1 \leq k \leq m$ such that for each k the actions of x_{2i-1} and x_{2i} are given by (1.4) or (1.5) with $[j_1, j_2, \ldots, j_\gamma]$ replaced by $[j_1, j_2, \ldots, j_\gamma]_k$. In particular, by (1A) such embedding of E in G is uniquely determined up to conjugacy in G. The proof of the remark is similar to that of the uniqueness of (1A) and Remark (2) of [5, (1D)].

(2) Suppose $r|q-\eta$, E has exponent r, and E is embedded in $\mathrm{GL}(r^\gamma, \eta q)$ as a subgroup. In the notation of (1A), we claim that V has an orthonormal basis $\{[j_1, j_2, \ldots, j_\gamma]'\}$, where $1 \leq j_i \leq r$ such that the actions of x_{2i-1} and x_{2i} for $i \geq 2$ are given by (1.4) with $[j_1, j_2, \ldots, j_\gamma]$ replaced by $[j_1, j_2, \ldots, j_\gamma]'$, and

$$x_1 \colon [j_1, j_2, \ldots, j_\gamma]' \mapsto [j_1 + 1, j_2, \ldots, j_\gamma]',$$
$$x_2 \colon [j_1, j_2, \ldots, j_\gamma]' \mapsto w^{-j_1}[j_1, j_2, \ldots, j_\gamma]'.$$

Indeed let $V_j' = \{v \in V \colon x_2 v = w^{-j} v\}$ for $1 \leq j \leq r$. Then V_j' are nondegenerate subspaces permuted by x_1 cyclically. If $\gamma = 1$ and $\{v_1\}$ is an orthonormal basis of V_1', then $\{[j_1]' = x_1^{j_1-1} v_1\}$, where $1 \leq j_1 \leq r$, is a required basis. Suppose $\gamma \geq 2$ and $\{[j_2, j_3, \ldots, j_\gamma]'\}$, where $1 \leq j_i \leq r$, is an orthonormal basis of V_1' such that the actions of $x_3, \ldots, x_{2\gamma}$ on the basis are given by (1.4) with $[j_2, j_3, \ldots, j_\gamma]$ replaced by $[j_2, j_3, \ldots, j_\gamma]'$. Let $[j_1, j_2, \ldots, j_\gamma]' = x_1^{j_1-1}[j_2, \ldots, j_\gamma]'$. Then $\{[j_1, j_2, \ldots, j_\gamma]' \colon 1 \leq j_i \leq r\}$ is a required basis.

(1B). *Suppose $r|q-\eta$. Let $G = \mathrm{GL}(r^\gamma, \eta q)$ and $R = ZE$ an r-subgroup of symplectic type of G, where $Z = Z(G)_r$ and E is an extraspecial subgroup of order $r^{2\gamma+1}$ of G. Set $C = C_G(R)$ and $N = N_G(R)$. Then $C = Z(G) = Z(N)$ and if E has exponent r, then $N/RC \simeq \mathrm{Sp}(2\gamma, q)$. In addition, if R is radical in G, then E has exponent r. Moreover, each linear character of $Z(N)$ acting trivially on $O_r(Z(N))$ has an extension to N trivial on R.*

Proof. By (1A) $\mathbb{F}_{q^2}$ is a splitting field, so that $C = Z(G) = Z(N)$. The proof of the last assertion is the same as that of [5, (1E)] with 2 replaced by r. If $R > E$, then E may be assumed to have exponent r. The elements of N induce automorphisms in $\mathrm{Aut}^0 E = \mathrm{Aut}^0 R$. Suppose E has exponent r and acts on the underlying space V of G by (1.4). We shall exhibit elements in N which together with R generate $\mathrm{Aut}^0 E$.

(1) Let g be the element in G such that

$$g \colon [j_1, j_2, \ldots, j_i, \ldots, j_\gamma] \mapsto [j_i, j_2, \ldots, j_1, \ldots, j_\gamma].$$

Then $g^{-1}x_1 g = x_{2i-1}$, $g^{-1}x_{2i-1}g = x_1$, $g^{-1}x_2 g = x_{2i}$, $g^{-1}x_{2i}g = x_2$, and $g^{-1}x_k g = x_k$ for all other indices. Thus N contains a subgroup inducing the symmetric group $\mathrm{S}(\gamma)$ on the set $\{E_1, E_2, \ldots, E_\gamma\}$.

(2) Let $\{[j_1, j_2, j_3, \ldots, j_\gamma]'\}$ be the orthonormal basis of V given by Remark (2), and g the element in G such that

$$g \colon [j_1, j_2, \ldots, j_\gamma]' \mapsto [j_1, j_2, \ldots, j_\gamma].$$

Then $g^{-1}x_1 g = x_2^{-1}$, $g^{-1}x_2 g = x_1$, and $g^{-1}x_k g = x_k$ for $k \geq 3$. By (1) for each $1 \leq i \leq \gamma$, there exists $h \in G$ such that $h^{-1}x_{2i-1}h = x_{2i}^{-1}$,

$h^{-1}x_{2i}h = x_{2i-1}$, and $h^{-1}x_k g = x_k$ for all other indices. Thus N contains a subgroup inducing Weyl group of type C_γ on $R/Z(R)$.

(3) Let g be the element in G such that

$$g\colon [j_1, j_2, j_3, \ldots, j_\gamma] \mapsto [\lambda j_1, j_2, j_3, \ldots, j_\gamma],$$

where λ is a nonzero element of $\mathbb{Z}/\mathbb{Z}r$. Then $g^{-1}x_1 g = x_1^\lambda$, $g^{-1}x_2 g = x_2^{\lambda^{-1}}$, and $g^{-1}x_k g = x_k$ for $k > 2$. In addition, let g be the element in G such that

$$g\colon [j_1, j_2, j_3, \ldots, j_\gamma] \mapsto [j_1 + j_2, j_2, j_3, \ldots, j_\gamma]. \tag{1.6}$$

Then $g^{-1}x_1 g = x_1 x_3$, $g^{-1}x_4 g = x_4 x_2^{-1}$, and $g^{-1}x_k g = x_k$ for all other indices. Since $\langle x_1, x_3, \ldots, x_{2\gamma-1}\rangle$ and $\langle x_2, x_4, \ldots, x_{2\gamma}\rangle$ give a hyperbolic decomposition of $R/Z(R)$, the element g of (1.6) induces

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & & & \\ & I & & \\ & & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \\ & & & I \end{pmatrix}$$

relative to this decomposition of $R/Z(R)$. By (1) we may replace E_1 and E_2 by E_i and E_j for $1 \le i \ne j \le \gamma$. Thus N contains a subgroup inducing

$$\left\langle \begin{pmatrix} A & \\ & (A^{-1})^t \end{pmatrix} : A \in \mathrm{GL}(\gamma, r) \right\rangle$$

on $R/Z(R)$.

(4) We claim there are elements in N inducing

$$\begin{pmatrix} I & X \\ & I \end{pmatrix}$$

on $R/Z(R)$ for any X such that $X^t = X$. By (3) it suffices to show this when

$$X = \mathrm{diag}\{1, 0, 0, \ldots, 0\}.$$

Indeed, let g be the element in G such that

$$g\colon [j_1, j_2, \ldots, j_\gamma] \mapsto w^{-(j_1+1)j_1/2}[j_1, j_2, \ldots, j_\gamma], \tag{1.7}$$

where w is the rth root of unity in $\mathbb{F}_{q^2}$ given by (1.4). Then $g^{-1}x_2 g = x_1 x_2$, and $g^{-1}x_k g = x_k$ for all other indices. Thus the claim holds.

By (3) and (4) N contains a subgroup inducing a Borel subgroup of $\mathrm{Sp}(2\gamma, r)$ on $R/Z(R)$. Thus N induces $\mathrm{Sp}(2\gamma, r)$ on $R/Z(R)$. Suppose R is radical in G. If E has exponent r^2, then $R = E$ and the element g defined by (1.7) lies in $N\backslash R$. Moreover, as shown in the proof of [20, p. 166], g induces an element of $Z(K)$, where $K \simeq \mathrm{Aut}^0 E/\mathrm{Inn}\, E$ is given by (1.1). Let $Q = \langle g, E\rangle$, so that $Q \le N$. We claim that $Q \le O_r(N)$. Indeed for any $h \in N$, h induces an element of $\mathrm{Aut}^0 E$. Replacing h by hx for some $x \in E$, we may suppose h induces an element of K. Thus $[h, g]$ induces a trivial action on E and then $[h, g] \in C = Z(G)$, so that $hgh^{-1} = zg$ for some $z \in C$ and $z \in O_r(C) = Z(R)$ since zg and g are r-elements. So h normalizes Q and the claim holds. It follows that R is nonradical in G and we may suppose E has exponent r. This proves (1B).

We now consider the embedding of R into finite classical groups. Let $G = \mathrm{U}(n, q)$, $\mathrm{Sp}(2n, q)$, $O(2n+1, q)$, or $\mathrm{O}^\eta(2n, q)$, and let V be the underlying space of G, where $\eta = \pm 1$. If V is a symplectic or orthogonal space, we always suppose the characteristic p of $\mathbb{F}_q$ is odd. Moreover, we denote by $I(V)$ the group of isometries of V, $I_0(V)$ the subgroups of $I(V)$ of determinant 1, and $\eta(V)$ the type of V if V is orthogonal. For simplicity, we set $\eta(V) = 1$ if V is symplectic.

We define the integers e, a, and sign $\varepsilon = \pm 1$ as follows: In the case $G = \mathrm{U}(n, q)$, let e be the order of $-q$ modulo r and $\varepsilon = 1$ or -1 according as e is even or odd; in the remaining cases, let e be the order of q^2 modulo r and ε the sign chosen so that r^a divides $q^e - \varepsilon$. In all cases, let r^a be the exact power of r dividing $q^{2e} - 1$. In the case $G = \mathrm{U}(n, q)$, our definition of e above is different from that of [11, p. 125]. In fact, if $r | q^e + 1$, then our e is the same as that of [11]. If $r | q^e - 1$, then our e is the double of that of [11].

We recall that there exists a set $\mathscr{F}$ of polynomials serving as elementary divisors for all semisimple elements of each of these groups. First suppose $G = \mathrm{U}(n, q)$. For each monic polynomial $\Delta(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0$ of $\mathbb{F}_{q^2}[X]$ with nonzero roots, let $\widetilde{\Delta}(X) = (a_0^{-1})^q X^m \Delta^q(X^{-1})$. Then define

$$\mathscr{F}_1 = \{\Delta \colon \Delta \text{ is monic, irreducible, } \Delta \neq X,\ \Delta = \widetilde{\Delta}\},$$
$$\mathscr{F}_2 = \{\Delta\widetilde{\Delta} \colon \Delta \text{ is monic, irreducible, } \Delta \neq X,\ \Delta \neq \widetilde{\Delta}\},$$

and $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$. Suppose G is a symplectic or orthogonal group. For each monic polynomial $\Delta(X)$ in $\mathbb{F}_q[X]$ with nonzero roots, let $\Delta(X)^*$ be the monic polynomial in $\mathbb{F}_q[X]$ whose roots are the inverses of the roots of $\Delta(X)$. Define

$$\mathscr{F}_0 = \{X - 1, X + 1\},$$
$$\mathscr{F}_1 = \{\Delta \colon \Delta \text{ is monic, irreducible, } \Delta \neq X,\ \Delta \neq X \pm 1, \text{ and } \Delta = \Delta^*\},$$
$$\mathscr{F}_2 = \{\Delta\Delta^* \colon \Delta \text{ is monic, irreducible, } \Delta \neq X,\ \Delta \neq X \pm 1, \text{ and } \Delta \neq \Delta^*\},$$

and $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2$. Given $\Gamma \in \mathscr{F}$, denote d_Γ its degree and δ_Γ its reduced degree defined by

$$\delta_\Gamma = \begin{cases} d_\Gamma & \text{if } G = \mathrm{U}(n, q) \text{ and } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2, \\ d_\Gamma & \text{if } G \neq \mathrm{U}(n, q) \text{ and } \Gamma \in \mathscr{F}_0, \\ \frac{1}{2} d_\Gamma & \text{if } G \neq \mathrm{U}(n, q) \text{ and } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2. \end{cases}$$

Thus δ_Γ is an integer. We define a sign ε_Γ for $\Gamma \in \mathscr{F}$ by

$$\varepsilon_\Gamma = \begin{cases} \varepsilon & \text{if } \Gamma \in \mathscr{F}_0, \\ -1 & \text{if } \Gamma \in \mathscr{F}_1, \\ 1 & \text{if } \Gamma \in \mathscr{F}_2. \end{cases}$$

Given a semisimple element $s \in G$, there exists a unique orthogonal decomposition

(1.8) $$V = \sum_\Gamma V_\Gamma(s), \qquad s = \prod_\Gamma s(\Gamma),$$

where the $V_\Gamma(s)$ are nondegenerate subspaces of V, $s(\Gamma)\in \mathrm{U}(V_\Gamma(s))$ or $I(V_\Gamma(s))$ according as V is or is not a unitary space, and $s(\Gamma)$ has minimal polynomial $\Gamma \in \mathscr{F}$. The decomposition (1.8) will be called the *primary* decomposition of s in G. Let $m_\Gamma(s)$ be the multiplicity of Γ in $s(\Gamma)$. Then

$$C_G(s) = \prod_\Gamma C_\Gamma(s), \tag{1.9}$$

where $C_\Gamma(s) = C_{\mathrm{U}(V_\Gamma(s))}(s(\Gamma))$ or $C_{I(V_\Gamma(s))}(s(\Gamma))$. Moreover, by [11, (1A)] or [12, (1.13)]

$$C_\Gamma(s) = \begin{cases} I(V_\Gamma(s)) & \text{if } \Gamma \in \mathscr{F}_0, \\ \mathrm{GL}(m_\Gamma(s), \varepsilon_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2. \end{cases} \tag{1.10}$$

A semisimple element $s \in G$ is *primary* if $s = s(\Gamma)$.

Suppose V is a symplectic or orthogonal space and s decomposes as (1.8). Let $\eta_\Gamma(s)$ be the type of $V_\Gamma(s)$, where $\eta_\Gamma(s) = 1$ for all $\Gamma \in \mathscr{F}$ if V is symplectic. So s lies in $I_0(V)$ if and only if $m_{X+1}(s)$ is even. By [12, (1.12)], the multiplicity and type functions $\Gamma \mapsto m_\Gamma(s)$, $\Gamma \mapsto \eta_\Gamma(s)$ satisfy the following relations

$$\begin{gathered} \dim V = \sum_\Gamma d_\Gamma m_\Gamma(s), \\ \eta(V) = (-1)^{(q-1)/2 m_{X-1}(s) m_{X+1}(s)} \prod_\Gamma \eta_\Gamma(s), \\ \eta(V_\Gamma(s)) = \varepsilon_\Gamma^{m_\Gamma(s)} \quad \text{for } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2, \text{ and } V \text{ orthogonal.} \end{gathered} \tag{1.11}$$

Conversely, if $\Gamma \mapsto \eta_\Gamma$, $\Gamma \mapsto \eta_\Gamma$ are functions from $\mathscr{F}$ to $\mathbb{N}$, $\{\pm 1\}$ respectively satisfying (1.11) with $m_\Gamma(s)$ and $\eta_\Gamma(s)$ replaced by n_Γ and η_Γ, then there exists a semisimple element s of $I(V)$ with those functions as multiplicity and type functions. Moreover, two semisimple elements s and s' of $I(V)$ are conjugate in $I(V)$ if and only if $m_\Gamma(s) = m_\Gamma(s')$ and $\eta_\Gamma(s) = \eta_\Gamma(s')$.

Let $Z = \langle z \rangle$ be a cyclic r-group of order $r^{a+\alpha}$ with $\alpha \geq 0$, E an extraspecial r-group of order $r^{2\gamma+1}$, and $R = ZE$ a group of symplectic type with $Z(R) = Z$. Moreover, we may suppose E has exponent r if $R > E$.

(1C). *Let $G = \mathrm{U}(n, q)$, $\mathrm{Sp}(2n, q)$, $O(2n+1, q)$, or $\mathrm{O}^\eta(2n, q)$, where $\eta = \pm 1$. Suppose $\mathbf{F}$ and $\mathbf{F}'$ are two embeddings of R in G such that $\mathbf{F}(z)$ and $\mathbf{F}'(z)$ are primary elements of G. Then $n = mer^{\alpha+\gamma}$ for some $m \geq 1$, $\mathbf{F}(R)$ and $\mathbf{F}'(R)$ are conjugate in G, and $\eta = \varepsilon^m$ if $G = \mathrm{O}^\eta(2n, q)$. Identify R with $\mathbf{F}(R)$ and let $C = C_G(R)$, $N = N_G(R)$, and $N^0 = \{g \in N : [g, Z] = 1\}$. Then $C \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$. Furthermore, suppose R is a radical subgroup of G.*

1. *E has exponent r and $N^0 = LC$, where $R \trianglelefteq L$, $L \cap C = Z(C) = Z(C_G(z)) = Z(L)$, $L/RZ(L) \simeq \mathrm{Sp}(2\gamma, r)$, and $[C, L] = 1$. Moreover, each linear character of $Z(L)$ acting trivially on $O_r(Z(L))$ can be extended as a character of L acting trivially on R.*
2. *$N/N^0 \simeq N_G(Z)/C_G(Z)$ is cyclic of order er^α or $2er^\alpha$ according as $G = \mathrm{U}(n, q)$ or $G \neq \mathrm{U}(n, q)$.*

Proof. Since both $Z(\mathbf{F}(R))$ and $Z(\mathbf{F}'(R))$ are cyclic groups of order $r^{a+\alpha}$ generated by primary elements $\mathbf{F}(z)$ and $\mathbf{F}'(z)$ respectively, they are conjugate in G, so that we may suppose $Z(\mathbf{F}(R)) = Z(\mathbf{F}'(R))$. Thus $\mathbf{F}(E)$ and $\mathbf{F}'(E)$

are subgroups of $C_G(\mathbf{F}(z))$. Let $H = C_G(\mathbf{F}(z))$ and Γ be the unique elementary divisor of $\mathbf{F}(z)$. Then $H \simeq \mathrm{GL}(m_\Gamma(\mathbf{F}(z)), \varepsilon q^{er^\alpha})$ and the two embeddings $\mathbf{F}(E)$ and $\mathbf{F}'(E)$ of E in H can be viewed as embeddings of E in $\mathrm{GL}(m_\Gamma(\mathbf{F}(z)), \varepsilon q^{er^\alpha})$ in which a generator y of $Z(E)$ is represented by scalar multiples of the identity matrix. It then follows by Remark (1) of (1A) that $\mathbf{F}(E)$ and $\mathbf{F}'(E)$ are conjugate in H and $m_\Gamma(\mathbf{F}(z)) = mr^\gamma$ for some $m \geq 1$. So $\mathbf{F}(R)$ and $\mathbf{F}'(R)$ are conjugate in G, and $\eta = \varepsilon^{mr^\gamma} = \varepsilon^m$ if $G = \mathrm{O}^\eta(2n, q)$.

Identify H with $\mathrm{GL}(mr^\gamma, \varepsilon q^{er^\alpha})$. Let $\mathbf{W}$ be the faithful and irreducible representation of E in $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ given by (1A), and let L_γ be the normalizer of $\mathbf{W}(E)$ in $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$. Then the commuting algebras of L_γ and E on the underlying space of $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ are $\mathbb{F}_{q^{er^\alpha}}$ or $\mathbb{F}_{q^{2er^\alpha}}$ according as $\varepsilon = 1$ or -1. Moreover, if E has exponent r, then $L_\gamma/Z(L_\gamma) \simeq \mathrm{Aut}^0 E$. By Remark (1) of (1A) $\mathbf{F}(E)$ in H can be viewed as an m-fold diagonal embedding of E into $\mathrm{GL}(mr^\gamma, \varepsilon q^{er^\alpha})$ given by

$$(1.12) \qquad \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \qquad g \in \mathbf{W}(E).$$

In particular, $C = C_H(\mathbf{F}(R)) \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$. Let L be the image of L_γ under (1.12), so that $\mathbf{F}(R) \trianglelefteq L$, $L \leq N^0 = N_H(\mathbf{F}(R))$, $C_H(L) = C_H(E) = C$, and $[L, C] = 1$. Suppose $\mathbf{F}(R)$ is radical in G and E has exponent r^2, so that $R = E$. As shown in the proof of (4) of (1B), there exists an r-element x of L_γ such that $x \notin \mathbf{W}(E)$ and x induces an element of $Z(\mathrm{Aut}^0 E/\mathrm{Inn}\, E)$, so that the image w of x under (1.12) is an r-element of $L \backslash \mathbf{F}(E)$. If $Q = \langle w, \mathbf{F}(E)\rangle$, then $C_H(\mathbf{F}(E)) = C_H(Q) = C$. Since $N^0 \trianglelefteq N$ and $\mathbf{F}(E)$ is radical in G, it follows that $\mathbf{F}(E) = O_r(N^0)$ and each element of N^0 induces an element of $\mathrm{Aut}^0 E$, so that w induces an element of $Z(\mathrm{Aut}^0 E/\mathrm{Inn}\, E)$. We claim $Q \leq O_r(N^0)$. Indeed for each $h \in N^0$, we may suppose h induces an element of $\mathrm{Aut}^0 E/\mathrm{Inn}\, E$ and then $[h, w]$ acts trivially on E, so that $[h, w] \in C$. Since h normalizes C and w commutes with C, $[h, w]$ commutes with C and $hwh^{-1} = gw$ for some $g \in Z(C) = Z(H)$. Since gw and w are commutative r-elements, g is an r-element of $Z(H)$, so that $g \in O_r(H) \leq \mathbf{F}(E)$. Thus h normalizes Q and $Q \leq O_r(N^0)$. This is a contradiction and E has exponent r.

Identify R with $\mathbf{F}(R)$. Since $L/Z(L) \simeq \mathrm{Aut}^0 R$ and N^0 induces a subgroup of $\mathrm{Aut}^0 R$, it follows that $N^0 = LC$. Thus $Z(H) \leq Z(N^0) \leq Z(L)Z(C)$, $Z(L) \leq Z(C) = Z(H)$, and $L \cap C \leq Z(C)$, so that $Z(L) = Z(H) = Z(C) = L \cap C$. The last assertion of (1) follows by (1B) since $L \simeq L_\gamma$. Finally, $N_G(Z)/C_G(Z)$ is cyclic of order er^α or $2er^\alpha$ according as $G = \mathrm{U}(n, q)$ or $G \neq \mathrm{U}(n, q)$ by [11, (3D)] or [12, (5B)]. Suppose g generates $N_G(Z)$ modulo $C_G(Z)$. Then E and $g^{-1}Eg$ are extraspecial subgroups of $H = C_G(Z)$, and they are conjugate in H by Remark (1) of (1A), so that $h^{-1}g^{-1}Egh = E$ for some $h \in H$ and $gh \in N$. On the other hand, $N \leq N_G(Z)$ and $N^0 = N \cap C_G(Z)$, so that $N/N^0 \simeq N_G(Z)/C_G(Z)$ and (1C) holds.

Remark. In the notation of (1C), let $E = \langle x_1, x_2, \dots, x_{2\gamma}\rangle$, $R' = \langle x_1, x_3, \dots, x_{2\gamma-1}, Z\rangle$. Identify R with $\mathbf{F}(R)$ and R' with $\mathbf{F}(R')$. Then $R' \trianglelefteq R$ and

$C_G(R') = C_1 \times C_2 \times \cdots \times C_{r^\gamma}$ is a regular subgroup G, where $C_i \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$ for all i. Indeed by Remark (1) of (1A) we may suppose the underlying space of $H = C_G(Z)$ has an orthonormal basis $\{[j_1, j_2, \ldots, j_\gamma]_k\}$, where $1 \le j_i \le r$ and $1 \le k \le m$, such that the actions of $x_1, x_2, \ldots, x_{2\gamma}$ on the basis are given by (1.4) or (1.5) with $[j_1, j_2, \ldots, j_\gamma]$ replaced by $[j_1, j_2, \ldots, j_\gamma]_k$. Thus each x_{2i-1} is a diagonal matrix with respect to the basis for $1 \le i \le \gamma$, so $C_H(R') = C_G(R') = C_1 \times C_2 \times \cdots \times C_{r^\gamma}$, where $C_i \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$ for all i.

2. The radical subgroups

In this section we shall give a description of the radical subgroups of classical groups. We first consider the unitary group $G = \mathrm{U}(n, q)$.

For integers $\alpha \ge 0$ and $\gamma \ge 0$, let Z_α be a cyclic group of order $r^{a+\alpha}$, E_γ an extraspecial group of order $r^{2\gamma+1}$, and $Z_\alpha E_\gamma$ a central product over $\Omega_1(Z_\alpha) = Z(E_\gamma)$. By (1A) $Z_\alpha E_\gamma$ can be embedded as a subgroup of $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ such that Z_α is identified with $O_r(Z(\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})))$. Let Λ_α be a polynomial in $\mathscr{F}$ having a primitive $r^{a+\alpha}$th root of unity as a root. The degree of Λ_α is er^α (cf. [11, p. 126]), so that $\mathrm{U}(er^{\alpha+\gamma}, q)$ has a primary element g with Λ_α as a unique elementary divisor of multiplicity r^γ. By (1.10)

$$C(g) \simeq \mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha}).$$

We may identify $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ with $C(g)$, so that $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ is embedded as a subgroup of $\mathrm{U}(er^{\alpha+\gamma}, q)$ and $Z_\alpha = \langle g \rangle$. Let $R_{\alpha,\gamma}$ be the image of $Z_\alpha E_\gamma$ under the composition

$$Z_\alpha E_\gamma \hookrightarrow \mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha}) \hookrightarrow \mathrm{U}(er^{\alpha+\gamma}, q).$$

Since $Z_\alpha = \langle g \rangle$, a generator of $Z(R_{\alpha,\gamma})$ is primary, so that by (1C) $R_{\alpha,\gamma}$ is uniquely determined by $Z_\alpha E_\gamma$ up to conjugacy. For integer $m \ge 1$, let $R_{m,\alpha,\gamma}$ be the image of the m-fold diagonal mapping of $R_{\alpha,\gamma}$ in $\mathrm{U}(mer^{\alpha+\gamma}, q)$ given by

$$g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \qquad g \in R_{\alpha,\gamma}.$$

Then a generator of $Z(R_{m,\alpha,\gamma})$ is the image of a generator of $Z(R_{\alpha,\gamma})$ under the embedding above, so that it is primary in $\mathrm{U}(mer^{\alpha+\gamma}, q)$ and then $R_{m,\alpha,\gamma}$ is uniquely determined by m and $Z_\alpha E_\gamma$ up to conjugacy. Let $C_{m,\alpha,\gamma}$ and $N_{m,\alpha,\gamma}$ be the centralizer and normalizer of $R_{m,\alpha,\gamma}$ in $\mathrm{U}(mer^{\alpha+\gamma}, q)$, and let $N^0_{m,\alpha,\gamma} = \{g \in N_{m,\alpha,\gamma} : [g, Z(R_{m,\alpha,\gamma})] = 1\}$. By (1C) $C_{m,\alpha,\gamma} \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma$, where I_γ is the identity matrix of order r^γ and $\mathrm{GL}(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma$ is the group $\{g \otimes I_\gamma : g \in \mathrm{GL}(m, \varepsilon q^{er^\alpha})\}$. If $R_{m,\alpha,\gamma}$ is radical, then E_γ has exponent r, $N^0_{m,\alpha,\gamma} = L_{m,\alpha,\gamma} C_{m,\alpha,\gamma}$, and $N_{m,\alpha,\gamma}/N^0_{m,\alpha,\gamma}$ is cyclic of order er^α, where $L_{m,\alpha,\gamma}$ is a subgroup of $N^0_{m,\alpha,\gamma}$ containing $R_{m,\alpha,\gamma}$ such that $L_{m,\alpha,\gamma} \cap C_{m,\alpha,\gamma} = Z(L_{m,\alpha,\gamma}) = Z(C_{m,\alpha,\gamma})$, $[L_{m,\alpha,\gamma}, C_{m,\alpha,\gamma}] = 1$, and $L_{m,\alpha,\gamma}/Z(L_{m,\alpha,\gamma}) R_{m,\alpha,\gamma} \simeq \mathrm{Sp}(2\gamma, r)$. In particular, $R_{m,\alpha,\gamma}$ is uniquely determined by m, α, and γ up to conjugacy. Moreover, each linear character of $Z(L_{m,\alpha,\gamma})$ acting trivially on $O_r(Z(L_{m,\alpha,\gamma}))$ can be extended as a character of $L_{m,\alpha,\gamma}$ trivial on $R_{m,\alpha,\gamma}$.

For integer $c \geq 1$, let A_c denote the elementary abelian r-subgroup of order r^c represented by its regular permutation representation. For any sequence $\mathbf{c} = (c_1, c_2, \ldots, c_l)$ of nonnegative integers, let $A_{\mathbf{c}} = A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_l}$, and let

$$R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_{\mathbf{c}}$$

be the wreath product in $\mathrm{U}(d, q)$, where $d = mer^{\alpha+\gamma+c_1+\cdots+c_l}$. Then $R_{m,\alpha,\gamma,\mathbf{c}}$ is determined up to conjugacy in $\mathrm{U}(d, q)$. By [3, (1.4)], which applies to $\mathrm{U}(d, q)$ with some obvious modifications,

$$C_{\mathrm{U}(d,q)}(R_{m,\alpha,\gamma,\mathbf{c}}) = C_{m,\alpha,\gamma} \otimes I_{\mathbf{c}}, \tag{2.1}$$

where $I_{\mathbf{c}}$ is the identity matrix of order $u = r^{c_1+c_2+\cdots+c_l}$ and $C_{m,\alpha,\gamma} \otimes I_{\mathbf{c}}$ is defined as before. Moreover,

$$\begin{aligned} &N_{\mathrm{U}(d,q)}(R_{m,\alpha,\gamma,\mathbf{c}}) = (N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \otimes N_{\mathbf{S}(u)}(A_{\mathbf{c}}), \\ &N_{\mathrm{U}(d,q)}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}} \simeq (N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r), \end{aligned} \tag{2.2}$$

where $(N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \otimes N_{\mathbf{S}(u)}(A_{\mathbf{c}})$ is defined as [3, (1.5)]. The proof of (2.2) is the same as that of [3, (4.1)] with GL replaced by U and some obvious modifications. We shall call $R_{m,\alpha,\gamma,\mathbf{c}}$ a *basic* subgroup of $\mathrm{U}(d, q)$, d the *degree* $d(R_{m,\alpha,\gamma,\mathbf{c}})$ of $R_{m,\alpha,\gamma,\mathbf{c}}$, and l the *length* $l(R_{m,\alpha,\gamma,\mathbf{c}})$ of $R_{m,\alpha,\gamma,\mathbf{c}}$.

Let V be a unitary space over $\mathbb{F}_{q^2}$, or a symplectic or orthogonal space over $\mathbb{F}_q$ with type $\eta = \pm 1$ if V is orthogonal. Let $G = \mathrm{U}(V)$ or $I(V)$, and let R be an r-subgroup of G. We shall say that an R-submodule W of V is *nondegenerate* or *totally isotropic* if W is respectively a nondegenerate or a totally isotropic subspace of V.

(2A). *Let R be an r-subgroup of G. Then V has an R-module decomposition*

$$V = V_1 \perp V_2 \perp \cdots \perp V_v \perp (U_{v+1} \oplus U'_{v+1}) \perp \cdots \perp (U_w \oplus U'_w), \tag{2.3}$$

where the V_i for $1 \leq i \leq v$ are nondegenerate simple R-submodules, the U_j and U'_j for $v+1 \leq j \leq w$ are totally isotropic simple R-submodules such that $U_j \oplus U'_j$ is nondegenerate and has no proper nondegenerate R-submodule. Moreover, if R is abelian and the set of vectors $[V, R]$ moved by R is V, then $v = 0$ or $v = w$ according as $\varepsilon = 1$ or -1.

Proof. The first half of (2A) follows by the proof of [5, (1B)]. Suppose R is abelian and $[V, R] = V$. Let $\mathbf{F}_i$ be the representation of R on V_i or $U_i \oplus U'_i$ according as $i \leq v$ or $i \geq v+1$. If $i \leq v$, then V_i is a simple R-module and the commuting algebra D of R on V_i contains $\mathbf{F}_i(R)$. If $i \geq v+1$, then U_i is a simple R-module and the representation of R on U'_i is the contragredient of the representation $\mathbf{W}$ of R on U_i composed with a field automorphism. Thus the commuting algebra D of R on U_i contains $\mathbf{W}(R)$. Since D is a field and $D^\times = D \backslash \{0\}$ is a cyclic group, $\mathbf{F}_i(R)$ is cyclic generated by g_i for some $g_i \in I(V_i)$ or $I(U_i \oplus U'_i)$ according as $i \leq v$ or $i \geq v+1$, so that V_i or U_i is a simple $\langle g_i \rangle$-module. By (1.8) g_i is primary with a unique elementary divisor $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ of multiplicity 1. Since g_i is an r-element, it follows that $\Gamma \in \mathscr{F}_1$ or $\mathscr{F}_2$ according as $\varepsilon = -1$ or 1. Thus the underlying space of $\mathbf{F}_i(R)$ has the form V_i or $U_i \oplus U'_i$ according as $\varepsilon = -1$ or 1. This proves (2A).

(2B). *Let* $G = \mathrm{U}(V)$, R *a radical* r*-subgroup of* G, *and* $N = N_G(R)$. *Then there exists a corresponding decomposition*

$$V = V_0 \perp V_1 \perp \cdots \perp V_t, \qquad R = R_0 \times R_1 \times \cdots \times R_t$$

such that R_0 *is the trivial subgroup of* $\mathrm{U}(V_0)$ *and* R_i *is a basic subgroup of* $\mathrm{U}(V_i)$ *for* $i \geq 1$. *Moreover, the extraspecial components of* R_i *for* $i \geq 1$ *have exponent* r.

Proof. Let $V_0 = C_V(R)$ be the set of vectors in V fixed by each element of R and $V_+ = [V, R]$. Then $V = V_0 \perp V_+$ and $R = R_0 \times R_+$, where $R_0 = \langle 1_{V_0} \rangle$ and $R_+ \leq \mathrm{U}(V_+)$. So $N = \mathrm{U}(V_0) \times N_{\mathrm{U}(V_+)}(R_+)$ and R_+ is necessarily radical in $\mathrm{U}(V_+)$. We may suppose $V = V_+$ by induction. Let $\mathbf{F}$ be the natural representation of R in G. The same proof with some obvious modifications as that of [5, (2B)] shows that R can be reduced to the following case: Every characteristic abelian subgroup of R is cyclic and $V = wV_1$ for some $w \geq 1$ such that either V_1 is a nondegenerate simple R-module or V_1 decomposes as $U_1 \otimes U_1'$, where U_1 and U_1' are totally isotropic simple R-modules and V_1 has no proper nondegenerate R-submodule. In particular, $Z(\mathbf{F}(R))$ is cyclic.

By a result of Hall, [14, 5.4.9], R is a group ZE of symplectic type, where Z is a cyclic r-group and E is an extraspecial r-group of order $r^{2\gamma+1}$. Thus $Z(\mathbf{F}(R)) = \mathbf{F}(Z)$ and we may suppose $\mathbf{F}(Z) = \langle z \rangle$. Let $H = C_G(\mathbf{F}(Z))$ and $C = C_G(\mathbf{F}(R))$. Then $\mathbf{F}(R) \leq H$ and $C \leq H$, so $Z(H) \trianglelefteq C$. Since $\mathbf{F}(R)$ is radical in G and $C \trianglelefteq N$, it follows $O_r(C) \leq Z(\mathbf{F}(R))$, so that $O_r(Z(H)) \leq O_r(C) \leq Z(\mathbf{F}(R))$ and $O_r(Z(C_G(z))) \leq \mathbf{F}(Z)$. Thus $O_r(Z(C_G(z)))$ is cyclic and by (1.9) and (1.10) z is primary with a unique elementary divisor $\Gamma \in \mathscr{F}$. So $H \simeq \mathrm{GL}(m_\Gamma(z), \varepsilon q^{\delta_\Gamma})$. Identify H with $\mathrm{GL}(m_\Gamma(z), \varepsilon q^{\delta_\Gamma})$. Then a generator of $\mathbf{F}(Z(E))$ is represented by a scalar multiple of the identity matrix, so that $m_\Gamma(z) = mr^\gamma$ for some integer $m \geq 1$ by Remark (1) of (1A). Since $O_r(Z(H)) \leq \mathbf{F}(Z)$ and $z \in O_r(Z(H))$, $\mathbf{F}(Z) = O_r(Z(H))$, so that $|Z| = r^{a+\alpha}$ for some integer $\alpha \geq 0$. By (1C) $R = R_{m,\alpha,\gamma}$ and E has exponent r. This proves (2B).

Let (R, φ) be a weight of $G = \mathrm{U}(V)$ and let

$$V = V_0 \perp V_1 \perp \cdots \perp V_t, \qquad R = R_0 \times R_1 \times \cdots \times R_t$$

be the corresponding decomposition of (2B). We define

$$V(m, \alpha, \gamma, \mathbf{c}) = \sum_i V_i, \qquad R(m, \alpha, \gamma, \mathbf{c}) = \prod_i R_i,$$

where i runs over all indices such that $R_i = R_{m,\alpha,\gamma,\mathbf{c}}$.

(2C). With the preceding notation

$$N(R) = \mathrm{U}(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{\mathrm{U}(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c})),$$

$$N(R)/R = \mathrm{U}(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{\mathrm{U}(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c}))/R(m, \alpha, \gamma, \mathbf{c}).$$

Moreover,

$$N_{\mathrm{U}(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c})) = N_{\mathrm{U}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u),$$

$$\begin{aligned} &N_{\mathrm{U}(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c}))/R(m, \alpha, \gamma, \mathbf{c}) \\ &\qquad = (N_{\mathrm{U}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u), \end{aligned}$$

where $V_{m,\alpha,\gamma,\mathbf{c}}$ is the underlying space of $R_{m,\alpha,\gamma,\mathbf{c}}$ and u is the number of basic components $R_{m,\alpha,\gamma,\mathbf{c}}$ in $R(m, \alpha, \gamma, \mathbf{c})$.

Proof. The proof of [3, (4B)] can be applied here with GL replaced by U and some obvious modifications.

We now consider radical subgroups of classical groups and as before, we suppose q is odd. For integers $\alpha \geq 0$ and $\gamma \geq 0$, let Λ_α be a polynomial in $\mathscr{F}$ having a primitive $r^{a+\alpha}$th root of unity as a root. Then the degree of Λ_α is $2er^\alpha$ and $\Lambda_\alpha \in \mathscr{F}_1$ or $\mathscr{F}_2$ according as $\varepsilon = -1$ or 1 (see [12, (5.1)]). Let $V_{\alpha,\gamma}$ be a symplectic or orthogonal space over $\mathbb{F}_q$ of dimension $2er^{\alpha+\gamma}$ and $\eta(V_{\alpha,\gamma}) = \varepsilon$ if $V_{\alpha,\gamma}$ is orthogonal. Then by (1.11) $I(V_{\alpha,\gamma})$ has a primary element g with a unique elementary divisor Λ_α of multiplicity r^γ. By (1.10) $C_{I(V_{\alpha,\gamma})}(g) \simeq \mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ and we may identify these two groups. By (1A) $Z_\alpha E_\gamma$ can be embedded as a subgroup of $\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})$ such that $Z_\alpha = O_r(Z(\mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha})))$, where $Z_\alpha E_\gamma$ is defined as before. The image $R_{\alpha,\gamma}$ of $Z_\alpha E_\gamma$ under the composition

$$Z_\alpha E_\gamma \hookrightarrow \mathrm{GL}(r^\gamma, \varepsilon q^{er^\alpha}) \hookrightarrow I(V_{\alpha,\gamma})$$

is then determined up to conjugacy. A generator of $Z(R_{\alpha,\gamma})$ is primary, so by (1C) $R_{\alpha,\gamma}$ is uniquely determined by $Z_\alpha E_\gamma$ up to conjugacy.

For integer $m \geq 1$, let $V_{m,\alpha,\gamma} = V_{\alpha,\gamma} \perp V_{\alpha,\gamma} \perp \cdots \perp V_{\alpha,\gamma}$ (m terms), and let $R_{m,\alpha,\gamma}$ be the image of the m-fold diagonal mapping of $R_{\alpha,\gamma}$ in $I(V_{m,\alpha,\gamma})$ given by

$$g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}, \qquad g \in R_{\alpha,\gamma}.$$

The same proof as the unitary case shows that $R_{m,\alpha,\gamma}$ is also uniquely determined by m and $Z_\alpha E_\gamma$ up to conjugacy. In addition, $\eta(V_{m,\alpha,\gamma}) = \varepsilon^m$ if $V_{m,\alpha,\gamma}$ is orthogonal.

Let $C_{m,\alpha,\gamma}$ and $N_{m,\alpha,\gamma}$ be the centralizer and normalizer of $R_{m,\alpha,\gamma}$ in $I(V_{m,\alpha,\gamma})$ respectively, and let $N^0_{m,\alpha,\gamma} = \{g \in N_{m,\alpha,\gamma} : [g, Z(R_{m,\alpha,\gamma})] = 1\}$. Then $N^0_{m,\alpha,\gamma} \trianglelefteq N_{m,\alpha,\gamma}$ and by (1C) $C_{m,\alpha,\gamma} \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma$, where I_γ is the identity matrix of degree r^γ and $\mathrm{GL}(m, \varepsilon q^{er^\alpha}) \otimes I_\gamma$ is defined as in the unitary case. In particular, if $R_{m,\alpha,\gamma}$ is radical in $I(V_{m,\alpha,\gamma})$, then $R_{m,\alpha,\gamma}$ has exponent r, $N^0_{m,\alpha,\gamma} = L_{m,\alpha,\gamma} C_{m,\alpha,\gamma}$, and $N_{m,\alpha,\gamma}/N^0_{m,\alpha,\gamma}$ is cyclic of order $2er^\alpha$, where $L_{m,\alpha,\gamma} \cap C_{m,\alpha,\gamma} = Z(L_{m,\alpha,\gamma}) = Z(C_{m,\alpha,\gamma})$, $[L_{m,\alpha,\gamma}, C_{m,\alpha,\gamma}] = 1$, $R_{m,\alpha,\gamma} \leq L_{m,\alpha,\gamma}$, and $L_{m,\alpha,\gamma}/Z(L_{m,\alpha,\gamma})R_{m,\alpha,\gamma} \simeq \mathrm{Sp}(2\gamma, r)$. So $R_{m,\alpha,\gamma}$ is uniquely determined by m, α, and γ up to conjugacy in $I(V_{m,\alpha,\gamma})$. Moreover, by (1C) each linear character of $Z(L_{m,\alpha,\gamma})$ acting trivially on $O_r(Z(L_{m,\alpha,\gamma}))$ can be extended as a character of $L_{m,\alpha,\gamma}$ acting trivially on $R_{m,\alpha,\gamma}$.

For each sequence $\mathbf{c} = (c_1, c_2, \dots, c_l)$ of nonnegative integers, let

$$(2.4) \qquad \begin{aligned} V_{m,\alpha,\gamma,\mathbf{c}} &= V_{m,\alpha,\gamma} \perp V_{m,\alpha,\gamma} \perp \cdots \perp V_{m,\alpha,\gamma} \qquad (u \text{ terms}), \\ A_{\mathbf{c}} &= A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_l}, \qquad R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_{\mathbf{c}}, \end{aligned}$$

where $u = r^{c_1+c_2+\cdots+c_l}$ and each A_{c_i} is defined as before. Then $R_{m,\alpha,\gamma,\mathbf{c}}$ is determined up to conjugacy in $I(V_{m,\alpha,\gamma,\mathbf{c}})$ and $\eta(V_{m,\alpha,\gamma,\mathbf{c}}) = \varepsilon^m$ if $V_{m,\alpha,\gamma,\mathbf{c}}$

is orthogonal. By [3, (1.4)] with some obvious modifications

$$C_{I(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}}) = C_{m,\alpha,\gamma} \otimes I_{\mathbf{c}},$$

where $I_{\mathbf{c}}$ is the identity matrix of order u and the right-hand sides is defined as before. Moreover, the same proof as that of [3, (4.1)] with GL replaced by I shows that

$$\begin{aligned} &N_{I(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}}) = (N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \otimes N_{\mathbf{S}(u)}(A_{\mathbf{c}}), \\ (2.5)\qquad &N_{I(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}} \\ &\quad = (N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r), \end{aligned}$$

where $(N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma}) \otimes N_{\mathbf{S}(u)}(A_{\mathbf{c}})$ is defined as [3, (1.5)]. We shall call $R_{m,\alpha,\gamma,\mathbf{c}}$ a *basic* subgroup of $I(V_{m,\alpha,\gamma,\mathbf{c}})$, $\dim V_{m,\alpha,\gamma,\mathbf{c}}$ the *degree* $d(R_{m,\alpha,\gamma,\mathbf{c}})$ of $R_{m,\alpha,\gamma,\mathbf{c}}$, and l the *length* $l(R_{m,\alpha,\gamma,\mathbf{c}})$ of $R_{m,\alpha,\gamma,\mathbf{c}}$.

(2D). *Let V be a symplectic or orthogonal space over $\mathbb{F}_q$, $G = I(V)$ the group of all isometries of V, and R a radical subgroup of G. Then there exists a corresponding decomposition*

$$V = V_0 \perp V_1 \perp \cdots \perp V_t, \qquad R = R_0 \times R_1 \times \cdots \times R_t,$$

such that R_0 is the trivial subgroup of $I(V_0)$ and R_i is a basic subgroup of $I(V_i)$ for $i \geq 1$. Moreover, the extraspecial components of R_i for $i \geq 1$ have exponent r.

Proof. Let $V_0 = C_V(R)$ and $V_+ = [V, R]$. Then $V = V_0 \perp V_+$ and $R = R_0 \times R_+$, where $R_0 = \langle 1_{V_0} \rangle$ and $R_+ \leq I(V_+)$. In particular, $N(R) = I(V_0) \times N_{I(V_+)}(R_+)$ and R_+ is necessarily a radical subgroup of $I(V_+)$. By induction we may suppose $V = V_+$. Thus $Z(R)$ is abelian and $[V, Z(R)] = V$. By (2A) we may write the $Z(R)$-module V as

$$V = m_1 V_1 \perp m_2 V_2 \perp \cdots \perp m_w V_w,$$

where each V_i is either a nondegenerate simple $Z(R)$-submodule or a sum $U_i \oplus U_i'$ of totally isotropic simple $Z(R)$-submodules U_i, U_i' according as $\varepsilon = -1$ or 1, and m_i is the multiplicity of V_i in V for all $i \geq 1$. If $\varepsilon = -1$, then $r | q^e + 1$ and $\mathbb{F}_{q^{2er^{\alpha_i}}}$ is the commuting algebra of $Z(R)$ on V_i for some $\alpha_i \geq 0$ since $[V_i, Z(R)] = V_i$ and $Z(R)$ is an r-group. Similarly, if $\varepsilon = 1$, then $r | q^e - 1$, $V_i = U_i \oplus U_i'$, and $\mathbb{F}_{q^{er^{\alpha_i}}}$ is the commuting algebra of $Z(R)$ on U_i for some integer $\alpha_i \geq 0$. In all cases $\dim V_i = 2er^{\alpha_i}$. Let $N^0 = \{g \in N(R) \colon [g, Z(R)] = 1\}$, and let $H = C_G(Z(R))$. Then $h(m_i V_i) = m_i V_i$ for $h \in H$ and all $i \geq 1$. Thus there exists a corresponding decomposition

$$H = H_1 \times H_2 \times \cdots \times H_w$$

such that $H_i \simeq \mathrm{GL}(m_i, \varepsilon q^{er^{\alpha_i}}) \leq I(m_i V_i)$ for all $i \geq 1$. Since R is radical and $N^0 \trianglelefteq N$, it follows $O_r(N^0) \leq O_r(N) = R$. On the other hand, $R \leq N^0$ and $N^0 = N_H(R)$, so $R = O_r(N^0)$ and R is radical in H.

Let R_i be the group of linear operators which agree with an element of R on $m_i V_i$ and are the identity on $m_j V_j$ for $j \neq i$. Then N^0 permutes the pairs $(m_i V_i, R_i)$ for $1 \leq i \leq w$, so that $R \leq N^0 \cap R_1 \times R_2 \times \cdots \times R_w \trianglelefteq N^0$. It follows that $R = R_1 \times R_2 \times \cdots \times R_w$ and $R_i = O_r(N_i)$, where $N_i = N_{H_i}(R_i)$. Thus R_i is radical in H_i for all i. By induction on $\dim V$, we may suppose $w = 1$, so that $V = m_1 V_1$, $R = R_1$, $H = H_1$, and $Z(R) = Z(R_1)$ is cyclic generated by some

$x \in I(V)$. But $H = C_G(x)$ and $O_r(Z(H)) \le O_r(H)$, so $O_r(Z(H)) \le Z(R)$. By (1.9) and (1.10) x is primary in G. Apply [3, (4A)] or (2B) to $H \simeq \mathrm{GL}(m_1, \varepsilon q^{er^{\alpha_1}})$. So R is a basic subgroup $R_{m,\alpha,\gamma,\mathbf{c}}$ of H, where m, γ, α are integers, and $\mathbf{c} = (c_1, \dots, c_l)$ is a sequence of nonnegative integers such that $\alpha \ge \alpha_1$, and $mer^{\alpha+\gamma+c_1+\cdots+c_l} = m_1 er^{\alpha_1}$. Moreover, the extraspecial components of $R_{m,\alpha,\gamma,\mathbf{c}}$ have exponent r. In particular, $\dim V = 2mer^{\alpha+\gamma+c_1+\cdots+c_l}$ and $\eta(V) = \varepsilon^m = \varepsilon^{m_1}$ if V is orthogonal. Thus $I(V)$ has a basic subgroup R' of the form $R_{m,\alpha,\gamma,\mathbf{c}}$ defined by (2.4), where the extraspecial components of R' have exponent r. So $Z(R)$ and $Z(R')$ are cyclic generated by primary elements of order $r^{a+\alpha}$ in $I(V)$, and they are conjugate in $I(V)$. Thus we may suppose $Z(R) = Z(R')$, so that $R' \le H$. By definition R' still has the type $R_{m,\alpha,\gamma,\mathbf{c}}$ as a subgroup of H, so that R' and R are conjugate in H. Thus $R = R_{m,\alpha,\gamma,\mathbf{c}}$ is a basic subgroup of $I(V)$ and (2D) follows.

Remark. In the notation of (2B) or (2D), suppose $t \ne 0$. Then there exists an element ρ of $Z(R)$ such that (1) $|\rho| = r^a$; (2) $[V, \rho] = \sum_{i=1}^t V_i$; (3) the restriction of ρ on $[V, \rho]$ is primary. Such an element exists by (2B) or (2D) and will be called a *primary* element of R. If ρ is a primary element of R, then $\langle \rho \rangle$ is uniquely determined by R up to conjugacy and $C_G(\rho) \simeq \mathrm{U}(V_0) \times \mathrm{GL}(m, \varepsilon q^e)$ or $C_G(\rho) \simeq I(V_0) \times \mathrm{GL}(m, \varepsilon q^e)$ for some $m \ge 1$ according as $G = \mathrm{U}(V)$ or $I(V)$.

Let (R, φ) be a weight of $G = I(V)$ and let

$$V = V_0 \perp V_1 \perp \cdots \perp V_t, \qquad R = R_0 \times R_1 \times \cdots \times R_t,$$

be the corresponding decomposition of (2D). We define

$$V(m, \alpha, \gamma, \mathbf{c}) = \sum_i V_i, \qquad R(m, \alpha, \gamma, \mathbf{c}) = \prod_i R_i,$$

where i runs over all indices such that $R_i = R_{m,\alpha,\gamma,\mathbf{c}}$.

(2E). *With the preceding notation*

$$N(R) = I(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{I(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c})),$$

$$N(R)/R = I(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{I(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c}))/R(m, \alpha, \gamma, \mathbf{c}).$$

Moreover,

$$N_{I(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c})) = N_{I(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u),$$

$$N_{I(V(m,\alpha,\gamma,\mathbf{c}))}(R(m, \alpha, \gamma, \mathbf{c}))/R(m, \alpha, \gamma, \mathbf{c}) = (N_{I(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}}) \wr \mathbf{S}(u),$$

where $V_{m,\alpha,\gamma,\mathbf{c}}$ is the underlying space of $R_{m,\alpha,\gamma,\mathbf{c}}$ and u is the number of basic components $R_{m,\alpha,\gamma,\mathbf{c}}$ in $R(m, \alpha, \gamma, \mathbf{c})$.

Proof. The proof is essentially the same as that of [3, (4B)] with GL replaced by I and some obvious modifications, except the minimal elements of $\mathscr{E}_i$ have dimension $2mer^{\alpha+\gamma}$ when $R_i = R_{m,\alpha,\gamma,\mathbf{c}}$, where $\mathscr{E}_i$ is defined there.

3. More on basic subgroups

Let R be a radical subgroup of a finite group G, $N = N(R)$, $C = C(R)$. The stabilizer in N of an irreducible character θ of CR will be denoted by

$N(\theta)$. We denote the sets of irreducible characters of $N(\theta)$ and N which cover θ and which have defect 0 as characters of $N(\theta)/R$ and N/R respectively by $\mathrm{Irr}^0(N(\theta), \theta)$ and $\mathrm{Irr}^0(N, \theta)$. By Clifford theory the induction mapping $\psi \mapsto I(\psi) = \mathrm{Ind}_{N(\theta)}^N(\psi)$ induces a bijection from $\mathrm{Irr}^0(N(\theta), \theta)$ to $\mathrm{Irr}^0(N, \theta)$. Since $\psi(1) = d(\psi)\theta(1)$ for some integral divisor $d(\psi)$ of $(N(\theta): CR)$, it follows that $(R, I(\psi))$ is a weight of G if and only if

$$d(\psi)_r = (N(\theta): CR)_r, \qquad \theta(1)_r = (CR: R)_r, \tag{3.1}$$

and in particular, θ then has defect 0 as a character of CR/R. In this case the block b of CR containing θ has a defect group R and the canonical character θ. Moreover, for any ψ of $\mathrm{Irr}^0(N(\theta), \theta)$, $I(\psi)$ is a character of b^N and $(R, I(\psi))$ is a b^G-weight of G. Following [3, p. 3] all B-weights for a block B of G have the form $(R, I(\psi))$, where R runs over representatives for the conjugacy G-classes of radical subgroups, b runs over representatives for the conjugacy $N(R)$-classes of blocks of $C(R)R$ such that b has defect group R and $b^G = B$, and ψ runs over $\mathrm{Irr}^0(N(\theta), \theta)$. Here θ is the canonical character of b. A pair (R, b) of an r-subgroup R of G and a block b of C is called a *Brauer pair* of G. In particular, pairs $(1, B)$ correspond to blocks B of G.

Now we consider the unitary group $G = \mathrm{U}(n, q)$. Given $\Gamma \in \mathscr{F}$, let $e_\Gamma, \alpha_\Gamma, m_\Gamma$ be integers defined as follows: e_Γ is the multiplicative order of $\varepsilon_\Gamma q^{d_\Gamma}$ modulo r, $r^{\alpha_\Gamma} = (d_\Gamma)_r$, and $m_\Gamma e r^{\alpha_\Gamma} = d_\Gamma e_\Gamma$. By [7, (3.2)] the Brauer pairs (R, b) of G are labeled by ordered triples (R, s, κ), where s is a semisimple r'-element of a dual group G^* of G, and $\kappa = \prod_{\Gamma \in \mathscr{F}} \kappa_\Gamma$ is a product of partitions κ_Γ such that each κ_Γ is an e_Γ-core of a partition of the multiplicity $m_\Gamma(s)$ of Γ in s. This labeling extends the labeling [11, (5D)] by Fong and Srinivasan for blocks B of G by ordered pairs (s, κ). Since $G^* \simeq G$, we may identify G^* with G.

Let $\mathscr{F}'$ be the subset of $\mathscr{F}$ consisting of polynomials whose roots have r'-order. In [11, (5A)] each Γ in $\mathscr{F}'$ determines a block B_Γ of $G_\Gamma = \mathrm{U}(e_\Gamma d_\Gamma, q)$ with defect group $R_\Gamma = R_{m_\Gamma, \alpha_\Gamma, 0}$ as follows: Let $C_\Gamma = C_{G_\Gamma}(R_\Gamma)$, $N_\Gamma = N_{G_\Gamma}(R_\Gamma)$, so that $C_\Gamma \simeq \mathrm{GL}(m_\Gamma, \varepsilon q^{e r^{\alpha_\Gamma}})$ and N_Γ / C_Γ is cyclic of order $e r^{\alpha_\Gamma}$. Then C_Γ has a block b_Γ with defect group R_Γ and label $(s_\Gamma, -)$ in C_Γ^* such that as an element of G_Γ^*, s_Γ is primary with a unique elementary divisor Γ of multiplicity e_Γ. If θ_Γ is the canonical character of b_Γ and $N(\theta_\Gamma)$ is its stabilizer in N_Γ, then $(N(\theta_\Gamma): C_\Gamma) = e_\Gamma$. The block b_Γ induces a block $b_\Gamma^{G_\Gamma}$ of G_Γ which will be denoted by B_Γ. Since $(e_\Gamma, r) = 1$, B_Γ has a defect group R_Γ and the label $(s_\Gamma, -)$ (see [7, 3.2]). We shall also write s_Γ as $e_\Gamma \Gamma$. Conversely, let $G = \mathrm{U}(m e r^\alpha, q)$, and B a block of G with defect group $R = R_{m, \alpha, 0}$. By [11, (4B) and (5A)] $(m, r) = 1$ and there exists a unique $\Gamma \in \mathscr{F}'$ such that Γ and B correspond in the preceding manner. In particular, $m = m_\Gamma$ and $\alpha = \alpha_\Gamma$.

The proofs of the following two lemmas are similar to that of [4, (3A) and (3B)].

(3A). *Given $\Gamma \in \mathscr{F}'$, let $G = \mathrm{U}(r^\gamma e_\Gamma d_\Gamma, q)$, $R = R_{m_\Gamma, \alpha_\Gamma, \gamma}$ a basic subgroup of G, and $C = C_G(R)$. Then $C = C_\Gamma \otimes I_\gamma$, where I_γ is the identity matrix of order r^γ. The irreducible character $\theta = \theta_\Gamma \otimes I_\gamma$ of C defined by $\theta(c \otimes I_\gamma) = \theta_\Gamma(c)$ for $c \in C_\Gamma$ is then a character of defect* 0 *of CR/R, and $|\mathrm{Irr}^0(N(\theta), \theta)| = e_\Gamma$.*

Proof. All statements but the last are clear. Let $N = N_G(R)$, and N^0 the subgroup $\{g \in N: [g, Z(R)] = 1\}$ of N. By (1C) $N^0 = LC$ and N/N^0 is cyclic of order er^{α_Γ}, where $R \leq L$, $L \cap C = Z(L) = Z(C)$, $[L, C] = 1$, and $L/Z(L)R \simeq \mathrm{Sp}(2\gamma, r)$. Moreover, each linear character of $Z(L)$ acting trivially on $O_r(Z(L))$ can be extended as a character of L trivial on R. Thus $N^0 \leq N(\theta)$, and $N(\theta)/N^0$ is cyclic. An irreducible constituent of the restriction of θ to $Z(C)$ is a linear character trivial on $O_r(Z(C))$ and so has an extension ξ to L trivial on R. Thus $\xi\theta$ is an extension of θ to N^0. Since $N^0/RC \simeq L/Z(L)R \simeq \mathrm{Sp}(2\gamma, r)$, the Steinberg character St of N^0/RC can be regarded as a character of N^0 trivial on CR. Thus $\vartheta = \mathrm{St}\,\xi\theta$ is irreducible since its restriction to C is irreducible. By (3.1) $\vartheta \in \mathrm{Irr}^0(N^0, \theta)$. Suppose ψ is a character of $\mathrm{Irr}^0(N^0, \theta)$. Then by Clifford theory $\psi = \chi\xi\theta$ for some irreducible character χ of N^0 trivial on C. Since ψ and $\xi\theta$ act trivially on R, χ acts trivially on R, so that χ is an irreducible character of N^0/CR. Since ψ has defect 0 as a character of N^0/R, χ has defect 0 as a character of $N^0/RC \simeq \mathrm{Sp}(2\gamma, r)$. Thus $\chi = \mathrm{St}$ and $\mathrm{Irr}^0(N^0, \theta) = \{\vartheta\}$. If $N(\vartheta)$ is the stabilizer of ϑ in N, then $N(\theta) = N(\vartheta)$ and $\mathrm{Irr}^0(N(\theta), \theta) = \mathrm{Irr}^0(N(\vartheta), \vartheta)$.

By (1C) a generator σ of N/N^0 induces a field automorphism of order er^{α_Γ} on $C(Z(R)) \simeq \mathrm{GL}(m_\Gamma r^\gamma, \varepsilon q^{er^{\alpha_\Gamma}})$. Since $C = C_\Gamma \otimes I_\gamma$ is a subgroup of $C(Z(R))$ invariant under σ, σ also induces a field automorphism of order er^{α_Γ} on C. But a generator σ_1 of N_Γ/C_Γ also induces a field automorphism of order er^{α_T} on $C_\Gamma \simeq \mathrm{GL}(m_\Gamma, \varepsilon q^{er^{\alpha_\Gamma}})$. By replacing generators, we may suppose σ induces σ_1 on C_Γ. It follows that $N(\theta)/N^0 \simeq N(\theta_\Gamma)/C_\Gamma$ and $|N(\theta)/N^0| = |N(\theta_\Gamma)/C_\Gamma| = e_\Gamma$. Since $N(\theta)/N^0$ is cyclic, ϑ extends in e_Γ ways to irreducible characters of $N(\vartheta)$ which cover ϑ, and since e_Γ is prime to r, these extensions are in $\mathrm{Irr}^0(N(\vartheta), \vartheta)$.

Remark. The weights $(R, I(\psi))$ of G for $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$ are B-weights, where B is the block of G labeled by $(r^\gamma e_\Gamma \Gamma, -)$, I is the induction operator from $N(\theta)$ to N, and $r^\gamma e_\Gamma \Gamma$ represents an element of $\mathrm{U}(r^\gamma e_\Gamma d_\Gamma, q)$ with a unique elementary divisor Γ of multiplicity $r^\gamma e_\Gamma$. Indeed, if b is the block of C containing θ, then (R, b) is labeled by $(R, r^\gamma e_\Gamma \Gamma, -)$ and the weights are b^G-weights. Moreover, by [7, 3.2] b^G is labeled by $(r^\gamma e_\Gamma \Gamma, -)$.

Given $\Gamma \in \mathscr{F}'$, let $G = \mathrm{U}(r^d e_\Gamma d_\Gamma, q)$ and $R = R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$ a basic subgroup of G, where d and γ are nonnegative integers, $\mathbf{c} = (c_1, c_2, \ldots, c_l)$ such that $\gamma + c_1 + c_2 + \cdots + c_l = d$. Then $C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, where $I_\gamma, I_{\mathbf{c}}$ are identity matrices of orders r^γ and $r^{c_1+c_2+\cdots+c_l}$ respectively. The irreducible character of C defined by

$$\theta(c \otimes I_\gamma \otimes I_{\mathbf{c}}) = \theta_\Gamma(c)$$

for $c \in C_\Gamma$ is then a character of defect 0 of CR/R. We shall say the pair (R, θ) is of *type* Γ. If b is the block of C containing θ, then (R, b) has label $(R, r^d e_\Gamma \Gamma, -)$.

(3B). *Let* $G = \mathrm{U}(n, q)$, R *a basic subgroup of* G, b *a block of* $C(R)R$ *with defect group* R, *and* θ *the canonical character of* b. *Then* (R, θ) *has type* Γ *for some* $\Gamma \in \mathscr{F}'$.

Proof. Suppose $R = R_{m,\alpha,\gamma,\mathbf{c}}$. Set $G_1 = \mathrm{U}(mer^\alpha, q)$, $R_1 = R_{m,\alpha,0}$, $C_1 = C_{G_1}(R_1)$. So $C_1 \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$ and $C = C_1 \otimes I_\gamma \otimes I_{\mathbf{c}}$. Then θ has the form

$\theta_1 \otimes I_\gamma \otimes I_{\mathbf{c}}$, where θ_1 is a character of C_1. Since θ has defect 0 as a character of CR/R and $CR/R \simeq C_1/R_1$, θ_1 also has defect 0 as a character of C_1/R_1. The block b_1 of C_1 containing θ_1 then has defect group R_1. By [11, (5A)] there is a unique $\Gamma \in \mathscr{F}'$ such that $R_1 = R_\Gamma$ and $\theta_1 = \theta_\Gamma$. Thus $m = m_\Gamma$, $\alpha = \alpha_\Gamma$, and (R, θ) has type Γ.

Following the notation of [12], we denote V and V^* finite-dimensional symplectic or orthogonal spaces over $\mathbb{F}_q$ related as follows:

(3.2)

V	$\dim V$	V^*	$\dim V^*$
symplectic	$2n$	orthogonal	$2n+1$
orthogonal	$2n+1$	symplectic	$2n$
orthogonal	$2n$	orthogonal	$2n$

where $\eta(V) = \eta(V^*) = 1$ in the first two cases and $\eta(V) = \eta(V^*)$ in the third case. Here $\eta(V) = 1$ for a symplectic space as before. Moreover, $I(V)$ and $I(V^*)$ are the groups of isometries of V and V^*, $I_0(V)$ and $I_0(V^*)$ the subgroups of $I(V)$ and $I(V^*)$ of determinant 1. We shall call $I_0(V^*)$ the *dual* group of $I_0(V)$. Let $G = I_0(V)$ and $G^* = I_0(V^*)$. Given a semisimple element s of G^*, let (s) be the conjugacy class of s in G^*, and let $\mathscr{E}(G, (s))$ be defined by [8, p. 57]. Namely $\mathscr{E}(G, (s))$ is the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with (s). Given a semisimple r'-element s of G^*, let

$$\mathscr{E}_r(G, (s)) = \bigcup_u \mathscr{E}(G, (su)),$$

where u runs over all the r-elements of $C_{G^*}(s)$. By [8, 2.2], $\mathscr{E}_r(G, (s))$ is a union of r-blocks.

The following lemma is due to Fong and Olsson.

(3C). *Let ρ be an r-element of G, b a block of $H = C_G(\rho)$, and B a block of G. Suppose H is regular subgroup of G, $B \subseteq \mathscr{E}_r(G, (s))$, and $b \subseteq \mathscr{E}_r(H, (t))$. If $b^G = B$, then s and t are conjugate in G^*.*

Proof. By Brauer's Second Main Theorem there exists a nonzero generalized decomposition number $d^\rho_{\chi,\varphi}$ for some irreducible character $\chi \in B$ and irreducible modular character $\varphi \in b$. Let $\chi^{(b')}(\rho\tau) = \sum_{\varphi' \in b'} d^\rho_{\chi,\varphi'}\varphi'(\tau)$, where b' is a block of H, τ runs over the r'-elements in H, and φ' runs over the irreducible modular characters in b'. Then $\chi(\rho\tau) = \sum_{b'} \chi^{(b')}(\rho\tau)$. On the other hand, by the theorem of Curtis type [9, (3.7)],

$$\chi(\rho\tau) = \sum_{b'} \sum_{\zeta \in b'} (\chi, R^G_H(\zeta))\zeta(\rho\tau),$$

where $R^G_H(\zeta)$ is the generalized Deligne-Lusztig character, b' runs over blocks of H, and ζ runs over the irreducible characters of b'. Since the $\zeta(\rho\tau)$ for $\zeta \in b'$ are linear combinations of the Brauer characters $\varphi(\tau)$ for $\varphi \in b'$ and the φ are linear independent, it follows that

$$\chi^{(b)}(\rho\tau) = \sum_{\zeta \in b} (\chi, R^G_H(\zeta))\zeta(\rho\tau),$$

and $\chi^{(b)}(\rho\tau) \neq 0$ for some r'-element τ. So $(\chi, R^G_H(\zeta)) \neq 0$ for some $\zeta \in b$. Suppose $\chi \in \mathscr{E}(G, (su))$ and $\zeta \in \mathscr{E}(H, (tv))$, where u is an r-element in

$C_{G^*}(s)$, v is an r-element in $C_{H^*}(t)$. Then su and tv are conjugate in G^*. Since s and t are the r'-parts of su and tv respectively, s and t are conjugate in G^*.

Let R be a radical r-subgroup of G, b a block of $C_G(R)R$ with defect group R, $V_0 = C_V(R)$, and $V_+ = [V, R]$. Then b^G is well defined and $b^G \subseteq \mathscr{E}_r(G, (s))$ for some $s \in G^*$. We shall give a decomposition of s corresponding to the decomposition $V_0 \perp V_+$ of V and give a label to the Brauer pair (R, b) when $V = V_+$, where b is regarded as a block of $C_G(R)$. Let ρ be a primary element of R given by the remark of (2D), and let $K = C_G(\rho)$. Then $K = K_0 \times K_+$, where $K_0 = I_0(V_0)$ and $K_+ \simeq \mathrm{GL}(m, \varepsilon q^e)$ for some $m \geq 0$. Since $\langle \rho \rangle \trianglelefteq R$, there exists a unique block B_ρ of K such that

$$(1, b^G) \leq (\langle \rho \rangle, B_\rho) \leq (R, b).$$

Let $B_\rho = B_{\rho,0} \times B_{\rho,+}$, where $B_{\rho,0}, B_{\rho,+}$ are blocks of K_0, K_+ respectively. Then $B_{\rho,0} \subseteq \mathscr{E}_r(K_0, (s_0))$ and $B_{\rho,+} \subseteq \mathscr{E}_r(K_+, (s_+))$ for some $s_0 \in K_0^*$ and $s_+ \in K_+^*$. By (3C) $s_0 \times s_+$ and s are conjugate in G^* and we may suppose $s = s_0 \times s_+$, so that this gives a decomposition of s. Moreover, the decomposition depends only on b^G not on the choice of R. Indeed there exists a defect group D of b^G such that $Z(D) \leq Z(R) \leq R \leq D$, so that $V_0 = C_V(D)$ and $V_+ = [V, D]$ and a primary element of D is a primary element of R. Thus we may suppose $\rho \in Z(D)$ is a primary element of D and then the decomposition $s = s_0 \times s_+$ is determined by b^G. Suppose now $V = V_+$. Then $B_\rho = B_{\rho,+}$ and $B_\rho \subseteq \mathscr{E}_r(K, (s))$. Since $C_G(R) = C_K(R)$, we may view (R, b) as a Brauer pair of K and then (R, b) has a Broué labeling $(R, t, -)$, where $t \in K^*$. Here, the third component of the label is empty since $K \simeq \mathrm{GL}(m, \varepsilon q^e)$ and R acts fixed-point freely on the underlying space of K. By definition of normal inclusion of Brauer pairs, $(1, B_\rho) \leq (R, b)$ holds in K and by [7, (3.2)], t and s are conjugate in K^*. In particular, t determines a unique conjugacy class of G^*. We then give (R, b) the label $(R, t, -)$.

Given Γ in $\mathscr{F}$, let e_Γ, α_Γ, and m_Γ be the following integers: e_Γ is the multiplicative order of $q^{2\delta_\Gamma}$ or $\varepsilon_\Gamma q^{\delta_\Gamma}$ modulo r according as $\Gamma \in \mathscr{F}_0$ or $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, $r^{\alpha_\Gamma} = (d_\Gamma)_r$, and $m_\Gamma e r^{\alpha_\Gamma} = \delta_\Gamma e_\Gamma$. In addition, let $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$.

Suppose $\dim V$ is even and s is a semisimple element of $I_0(V^*)$ with primary decomposition

$$V^* = \sum_\Gamma V_\Gamma^*(s), \qquad s = \prod_\Gamma s(\Gamma). \tag{3.3}$$

We define a semisimple element s^* of $I_0(V)$, which is determined uniquely up to conjugacy in $I(V)$, as follows: If V is orthogonal, then V and V^* have the same dimension and type, so that $m_\Gamma(s)$ and $\eta_\Gamma(s)$ satisfy the relations (1.11). Thus a semisimple element, denoted by s^*, exists in $I(V)$ such that $m_\Gamma(s^*) = m_\Gamma(s)$ and $\eta_\Gamma(s^*) = \eta_\Gamma(s)$. Since $s \in G^*$, it follows that $s^* \in G$. If V is symplectic, then V^* is an odd dimensional orthogonal space. Let $\eta_\Gamma = 1$ for all $\Gamma \in \mathscr{F}$, and $n_\Gamma = m_\Gamma(s)$ except when $\Gamma = X - 1$, in which case, $n_\Gamma = m_\Gamma(s) - 1$. Then n_Γ and η_Γ satisfy the relations (1.11) with $m_\Gamma(s)$ and $\eta_\Gamma(s)$ replaced by n_Γ and η_Γ respectively. So a semisimple element, denote by s^*, exists in G such that $m_\Gamma(s^*) = n_\Gamma$ and $\eta_\Gamma(s^*) = \eta_\Gamma = 1$. Thus s^* is uniquely determined up to conjugacy in $I(V)$ and $\det s^* = 1$. We shall call s^* a *dual* of s.

The following proposition is due to Fong and Olsson.

(3D). *The dual mapping $s \mapsto s^*$ induces a bijection $f\colon (s) \mapsto (s^*)$ from the conjugacy classes of r-elements of $I_0(V^*)$ onto the conjugacy classes of r-elements of $I_0(V)$ such that*

$$C_{I_0(V)}(s^*) \simeq C_{I_0(V^*)}(s)^*. \tag{3.4}$$

Proof. Suppose s is an r-element and decomposes as (3.3). Then -1 is not an eigenvalue of s, so that $\dim V^*_\Gamma(s) = m_\Gamma(s)\, d_\Gamma$ and $\eta_\Gamma(s) = \varepsilon^{m_\Gamma(s)}$ for $\Gamma \neq X-1$. Thus

$$m_{X-1}(s) = \dim V^* - \sum_{\Gamma \neq X-1} \dim V^*_\Gamma(s)$$

and

$$\eta_{X-1}(s) = (-1)^{(q-1)/2 m_{X-1}(s) m_{X+1}(s)} \eta(V^*) \prod_{\Gamma \neq X-1} \eta_\Gamma(s),$$

so that s is determined uniquely up to conjugacy in $I(V^*)$ by its multiplicity function $m_\Gamma(s)$. Moreover, $s \in I_0(V^*)$ and the $I(V^*)$-class of s decomposes into one or two conjugacy classes of $I_0(V^*)$ according as 1 is or is not an eigenvalue of s. Similar statements hold for r-elements of $I(V)$. If V is symplectic, then the dual mapping induces a bijection of the conjugacy classes of r-elements of $I_0(V^*)$ onto the conjugacy classes of r-elements of $I_0(V)$. If V and V^* are even dimensional orthogonal spaces, then the dual mapping induces a bijection of the conjugacy classes of r-elements of $I(V^*)$ onto the conjugacy classes of r-elements of $I(V)$. Moreover, the $I(V^*)$-class of s is a single $I_0(V^*)$-class if and only if the $I(V)$-class of s^* is a single $I_0(V)$-class. So the dual mapping induces a bijection of the conjugacy classes of r-elements of $I_0(V^*)$ and $I_0(V)$. The isomorphism (3.4) follows by [12, (3A)].

Given $m \geq 1$, let V be a symplectic or orthogonal space of dimension $2em$ and type ε^m if V is orthogonal. Let $G = I_0(V)$ and $G^* = I_0(V^*)$. By [12, (5.2)] G has a basic subgroup R of the form $R_{m,0,0}$, and we denote by u^* a primary element of R and u a dual of u^* given by (3D), so that $|u^*| = r^a$, $u^* = u^*(\Gamma)$ for a unique $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, and $C_G(u^*) = C_{I(V)}(u^*) \simeq \mathrm{GL}(m, \varepsilon q^e)$. Moreover, the subgroup $\langle u^* \rangle$ is uniquely determined up to conjugacy in $I(V)$. Namely, if $v^* \in G$ is an element of order r^a and $v^* = v^*(\Gamma')$ for a unique $\Gamma' \in \mathscr{F}_1 \cup \mathscr{F}_2$, then $\langle v^* \rangle$ and $\langle u^* \rangle$ are conjugate in $I(V)$. Let $\mathscr{S}$ and $\mathscr{S}^*$ be the sets of conjugacy classes of G and G^* of semisimple elements in the r-sections containing u^* and u respectively. Here the r-section containing u^* in G, by definition, is the set of all elements in G whose r-part is conjugate with u^* in G. Thus each class of $\mathscr{S}$ has the form $(h^* u^*)$ for some semisimple r'-element $h^* \in C_G(u^*)$. Define

$$\mathscr{S}' = \{[h^*] \colon (h^* u^*) \in \mathscr{S}\}, \qquad \mathscr{S}^{*\prime} = \{[s] \colon (su) \in \mathscr{S}^*\},$$

where $[h^*]$ and $[s]$ are conjugacy classes of h^* and s in $I(V)$ and $I(V^*)$ respectively.

(3E). *The dual mapping* $s \mapsto s^*$ *from the semisimple elements of* $I_0(V^*)$ *to the semisimple elements of* $I_0(V)$ *induces a bijection* $f\colon [s] \mapsto [s^*]$ *from* $\mathscr{S}^{*\prime}$ *onto* $\mathscr{S}'$ *such that*

$$C_{I_0(V)}(u^*, s^*) \simeq C_{I_0(V^*)}(u, s). \tag{3.5}$$

Proof. Let $[s] \in \mathscr{S}^{*\prime}$, s^* a dual of s in G, $K = C_G(u^*)$, and $K^* = C_{G^*}(u)$, so that K^* is a dual of K. Then s and s^* have primary decompositions

$$V = \sum_\Gamma V_\Gamma(s^*), \quad s^* = \prod_\Gamma s^*(\Gamma), \quad V^* = \sum_\Gamma V_\Gamma^*(s), \quad s = \prod_\Gamma s(\Gamma). \tag{3.6}$$

Thus $C_{I(V^*)}(s) = \prod_\Gamma C_\Gamma(s)$, where $C_\Gamma(s) = C_{I(V_\Gamma^*(s))}(s(\Gamma)$. Moreover, by (1.10)

$$C_\Gamma(s) \simeq \begin{cases} I(V_\Gamma^*(s)) & \text{if } \Gamma \in \mathscr{F}_0, \\ \mathrm{GL}(m_\Gamma(s), \varepsilon_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2. \end{cases} \tag{3.7}$$

Let u_Γ be the restriction of u to $V_\Gamma^*(s)$. Then $[V_\Gamma^*(s), u_\Gamma] = V_\Gamma^*(s)$ for $\Gamma \neq X - 1$ and $u_\Gamma \in C_\Gamma(s)$. Thus

$$m_\Gamma(s) = \begin{cases} e_\Gamma w_\Gamma(s) & \text{if } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2, \\ 2ew_\Gamma(s) & \text{if } \Gamma \in \mathscr{F}_0 \text{ and } \dim V^* \text{ is even}, \\ 2ew_\Gamma(s) & \text{if } \Gamma = X+1 \text{ and } \dim V^* \text{ is odd}, \\ 2ew_\Gamma(s)+1 & \text{if } \Gamma = X-1 \text{ and } \dim V^* \text{ is odd}, \end{cases} \tag{3.8}$$

for some integer $w_\Gamma(s)$, and $\eta_{X+1}(s) = \varepsilon^{w_{X+1}(s)}$, $\eta_\Gamma(s) = \varepsilon_\Gamma^{m_\Gamma(s)}$ for $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$. Moreover, $\eta_{X-1}(s)$ is determined by the equation

$$\eta(V^*) = (-1)^{(q-1)/2 m_{X-1}(s) m_{X+1}(s)} \prod_\Gamma \eta_\Gamma(s). \tag{3.9}$$

Thus the type function $\eta_\Gamma(s)$ is uniquely determined by the multiplicity function $m_\Gamma(s)$, so that $[s] = [s']$ for $[s], [s'] \in \mathscr{S}^{*\prime}$ if and only if $m_\Gamma(s) = m_\Gamma(s')$ for all $\Gamma \in \mathscr{F}$. It is clear that $C_{K^*}(s) = C_{G^*}(u, s) = C_{C_{G^*}(s)}(u)$ and $C_{K^*}(s) = \prod_\Gamma C_\Gamma(u, s)$, where $C_\Gamma(u, s) = C_{C_\Gamma(s)}(u_\Gamma)$ for $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $C_\Gamma(u, s) = C_{I_0(V_\Gamma^*(s))}(u_\Gamma)$ for $\Gamma \in \mathscr{F}_0$. By (3.7) and (3.8)

$$C_\Gamma(u, s) \simeq \mathrm{GL}(w_\Gamma(s), \varepsilon_\Gamma q^{e_\Gamma \delta_\Gamma}) \tag{3.10}$$

for all $\Gamma \in \mathscr{F}$. Similarly, $C_{I(V)}(s^*) = \prod_\Gamma C_\Gamma(s^*)$, where $C_\Gamma(s^*) = C_{I(V_\Gamma(s^*))}(s^*(\Gamma))$. Moreover,

$$C_\Gamma(s^*) = \begin{cases} I(V_\Gamma(s^*)) & \text{if } \Gamma \in \mathscr{F}_0, \\ \mathrm{GL}(m_\Gamma(s^*), \varepsilon_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2. \end{cases}$$

By definition of s^*, $m_\Gamma(s^*) = m_\Gamma(s)$ except when $\Gamma = X - 1$ and V is symplectic, in which case, $m_\Gamma(s^*) = m_\Gamma(s) - 1$. Thus $m_\Gamma(s^*) = \beta_\Gamma e_\Gamma w_\Gamma(s)$, where $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$. Let $w_\Gamma(s) = \sum_\beta n_\beta r^\beta$ be the r-adic expansion of $w_\Gamma(s)$, and $\mathbf{c}_\beta = (1, 1, \dots, 1)$ (β terms). Then a Sylow r-subgroup $D(\Gamma)$ of $C_\Gamma(s^*)$ is of the form $\prod_\beta (R_{m_\Gamma, \alpha_\Gamma, 0, \mathbf{c}_\beta})^{n_\beta}$. Thus a Sylow r-subgroup P of $C_{I(V)}(s^*)$ is of the form $\prod_\Gamma D(\Gamma)$ as a subgroup of $I(V)$, so that P has a primary element v^* and $\langle v^* \rangle$ is conjugate with $\langle u^* \rangle$ in $I(V)$. Thus a conjugate of s^* in $I(V)$ lies in K. Replacing s^* by its conjugate, we may suppose $s^* \in K$. So $C_K(s^*) = C_G(u^*, s^*) = C_{C_G(s^*)}(u^*)$ and if u^*_Γ is the restriction of u^* to $V_\Gamma(s^*)$, then $C_K(s^*) = \prod_\Gamma C_\Gamma(u^*, s^*)$, where $C_\Gamma(u^*, s^*) = C_{C_\Gamma(s^*)}(u^*_\Gamma)$. Moreover,

$$C_\Gamma(u^*, s^*) \simeq \mathrm{GL}(w_\Gamma(s), \varepsilon_\Gamma q^{e_\Gamma \delta_\Gamma}), \tag{3.11}$$

for all $\Gamma \in \mathscr{F}$. Since s^* is an r'-element and $s^* \in K$, it follows $(s^* u^*) \in \mathscr{S}$ and $[s^*] \in \mathscr{S}'$.

Conversely, given $[s^*] \in \mathscr{S}'$, suppose s^* decomposes as (3.6). Since $u^* \in C_G(s^*)$ and the restriction u^*_Γ of u^* to $V_\Gamma(s^*)$ lies in $C_\Gamma(s^*)$, it follows $m_\Gamma(s^*) = \beta_\Gamma e_\Gamma w_\Gamma(s^*)$. Define $n_\Gamma = m_\Gamma(s^*)$ except when $\Gamma = X - 1$ and V is symplectic, in which case, $n_\Gamma = m_\Gamma(s^*) + 1$. In addition, define $\eta_\Gamma = \varepsilon_\Gamma^{m_\Gamma(s^*)}$ for $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, $\eta_{X+1} = \varepsilon^{w_{X+1}(s^*)}$, and η_{X-1} is chosen so that (3.9) holds with $\eta_\Gamma(s)$ and $m_\Gamma(s)$ replaced by η_Γ and n_Γ respectively. Thus n_Γ and η_Γ satisfy the relation (1.11) for V^* with $m_\Gamma(s)$ and $\eta_\Gamma(s)$ replaced by n_Γ and η_Γ, so that a semisimple element, denote by s, exists in $I_0(V^*)$ such that $m_\Gamma(s) = n_\Gamma$ and $\eta_\Gamma(s) = \eta_\Gamma$. Such an element is determined uniquely up to conjugacy in $I(V^*)$. Thus $m_\Gamma(s)$ satisfy equation (3.8) with $w_\Gamma(s)$ replaced by $w_\Gamma(s^*)$. A similar proof to above shows that a Sylow r-subgroup of $C_{I(V^*)}(s)$ has a primary element conjugate with u in $I(V^*)$. We may suppose $u \in C_{I(V^*)}(s)$ and $(su) \in \mathscr{S}^*$, so that $[s] \in \mathscr{S}^{*\prime}$. But $[s] = [s']$ for $[s], [s'] \in \mathscr{S}^*$ if and only if $m_\Gamma(s) = m_\Gamma(s')$ for all $\Gamma \in \mathscr{F}$, so the two maps induced by $s \mapsto s^*$ and $s^* \mapsto s$ are inverse each other and both are bijections. The isomorphism (3.5) follows by (3.10) and (3.11).

Remark. As shown in the proof of (3E), if s^* is a semisimple r'-element of $I_0(V)$ such that a Sylow r-subgroup of $C_{I(V)}(s^*)$ acts fixed-point freely on V, then $m_\Gamma(s^*) = \beta_\Gamma e_\Gamma w_\Gamma(s^*)$, so that a dual s of s^* is a well-defined semisimple r'-element of $I_0(V^*)$. Moreover, if u^* is a primary element of a Sylow r-subgroup of $C_{I(V)}(s^*)$ and u is its dual, then we may suppose u is a primary element of a Sylow r-subgroup of $C_{I(V^*)}(s)$ and $C_{I_0(V)}(u^*, s^*) \simeq C_{I_0(V^*)}(u, s)$.

(3F). *Given integer $m \geq 1$, let V be a symplectic or orthogonal space over $\mathbb{F}_q$ of dimension $2em$ and $\eta(V) = \varepsilon^m$ if V is orthogonal. Let $G = I_0(V)$, and B a block of G contained in $\mathscr{E}_r(G, (s))$ for some semisimple r'-element s of G^*. If a defect group R of B acts fixed-point freely on V, then R is conjugate in $I(V)$ with a Sylow r-subgroup of $C_G(s^*)$, where s^* is a dual of s in G.*

Proof. Since R is radical in $I(V)$, it has a primary element z^*. Let $K = C_G(z^*)$ and K^* its dual. Then $z^* = z^*(\Gamma)$ for a unique $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, $K = C_{I(V)}(z^*) \simeq \mathrm{GL}(m, \varepsilon q^e)$, and K^* is embedded as a regular subgroup in G^*. Suppose (z^*, B_{z^*}) is a major subsection associated with B, in the sense of [6], and $B_{z^*} \subseteq \mathscr{E}_r(K, (t))$. Then s and t are conjugate in G^* by (3C) and R is a defect group of B_{z^*}. Replace s by a conjugate we may suppose $s = t$, so that R is conjugate with a Sylow r-subgroup of $C_{K^*}(s)^*$ in K by a result of [11, §5]. Let s^* be a dual of s and ρ an element of order r^a in $Z(K^*)$. Such an element ρ exists since $K \simeq K^*$. Thus $K^* \leq C_{G^*}(\rho)$ and $\delta_\Gamma = e$ for all $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ with $m_\Gamma(\rho) \neq 0$. By (1.9) and (1.10) $C_{G^*}(\rho) = K^*$, so that ρ is a primary element of $O_r(Z(K^*))$. Thus $\langle \rho \rangle$ is conjugate in $I(V^*)$ with the subgroup generated by a dual of z^* given by (3D). Replacing ρ by ρ^k for some integer k, we may suppose ρ is a dual of z^*. Since s lies in the r-section containing ρ, we may suppose s^* lies in the r-section containing z^*

and $C_K(s^*) \simeq C_{K^*}(s)$ by (3E). By (3.10) and (3.11) $C_K(s^*)$ and $C_{K^*}(s)^*$ are conjugate in K. Thus R is conjugate with a Sylow r-subgroup of $C_K(s^*)$.

We may suppose R is a Sylow r-subgroup of $C_K(s^*)$. Let P be a Sylow r-subgroup of $C_{I(V)}(s^*)$ containing R and u^* a primary element of P. So $u^* \in Z(P)$, $R \leq C_P(z^*) \leq C_K(s^*)$, and $R = C_P(z^*)$ since R is Sylow in $C_K(s^*)$. Thus $u^* \in Z(R)$ and u^* is a primary element of R. So $\langle z^* \rangle = \langle u^* \rangle \leq Z(P)$, $P = C_P(z^*) = R$, and (3F) holds.

Let $\mathscr{F}'$ be the subsets of polynomials in $\mathscr{F}$ whose roots have r'-order. Given $\Gamma \in \mathscr{F}'$, we shall define G_Γ, R_Γ, C_Γ, θ_Γ, and s_Γ as follows: Let V_Γ denote a symplectic or orthogonal space of dimension $2e_\Gamma\delta_\Gamma$ over $\mathbb{F}_q$ and of type $\varepsilon_\Gamma^{e_\Gamma}$ or ε according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$ if V_Γ is orthogonal. Thus $I(V_\Gamma)$ has a primary element s_Γ^* with a unique elementary divisor Γ of multiplicity of $\beta_\Gamma e_\Gamma$ and $I(V_\Gamma)$ has a basic subgroup R_Γ of form $R_{m_\Gamma, \alpha_\Gamma, 0}$ by [12, (1.12) and (5.2)]. Let $G_\Gamma = I(V_\Gamma)$, $G_\Gamma^0 = I_0(V_\Gamma)$, and $C_\Gamma = C_{G_\Gamma}(R_\Gamma)$. Then $s_\Gamma^* \in G_\Gamma^0$ and $C_\Gamma \simeq \mathrm{GL}(m_\Gamma, \varepsilon q^{e r^{\alpha_\Gamma}})$, so that a Coxeter torus T_Γ of C_Γ has order $q^{m_\Gamma e r^{\alpha_\Gamma}} - \varepsilon^{m_\Gamma}$. The dual T_Γ^* is embedded as a regular subgroup of C_Γ^*, and in turn, C_Γ^* is embedded as a regular subgroup of G_Γ^{0*}. We claim that there exists an element s_Γ in T_Γ^* such that $C_{C_\Gamma^*}(s_\Gamma) = T_\Gamma^*$ and as an element of G_Γ^{0*}, s_Γ and s_Γ^* are dual each other in the sense of (3E). Indeed

$$C_{G_\Gamma}(s_\Gamma^*) = \begin{cases} I(V_\Gamma) & \text{if } \Gamma = X \pm 1, \\ \mathrm{GL}(e_\Gamma, \varepsilon_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \neq X \pm 1, \end{cases}$$

so that a Sylow r-subgroup of $C_{G_\Gamma}(s_\Gamma^*)$ acts fixed-point freely on V_Γ. By the remark of (3E) a dual s_Γ of s_Γ^* exists in G_Γ^{0*} and

$$C_{G_\Gamma^{0*}}(s_\Gamma) = \begin{cases} \mathrm{GL}(e_\Gamma, \varepsilon_\Gamma q^{\delta_\Gamma}) & \text{if } \Gamma \neq X \pm 1, \\ \mathrm{SO}^\varepsilon(2e, q) & \text{if } \Gamma = X \pm 1 \text{ and } V_\Gamma \text{ is orthogonal}, \\ \langle w, 1 \times \mathrm{SO}^\varepsilon(2e, q) \rangle & \text{if } \Gamma = X + 1 \text{ and } V_\Gamma \text{ is symplectic}, \\ \mathrm{SO}(2e+1, q) & \text{if } \Gamma = X - 1 \text{ and } V_\Gamma \text{ is symplectic}, \end{cases}$$

where w is an element in $\mathrm{SO}(V_\Gamma^*)$ such that $w^2 \in 1 \times \mathrm{SO}^\varepsilon(2e, q)$, and 1 is the identity matrix of size 1. Let $R_\Gamma'^*$ be a Sylow r-subgroup of $C_{G_\Gamma^{0*}}(s_\Gamma)$, $C_\Gamma'^* = C_{G_\Gamma^{0*}}(R_\Gamma'^*)$, and $T_\Gamma'^* = C_{C_\Gamma'^*}(s_\Gamma)$. Then $s_\Gamma \in T_\Gamma'^*$ and $T_\Gamma'^* = C_{C_{G_\Gamma^{0*}}(s_\Gamma)}(R_\Gamma'^*)$. Thus $T_\Gamma'^*$ has order $q^{e_\Gamma\delta_\Gamma} - \varepsilon_\Gamma^{e_\Gamma}$. But $e_\Gamma\delta_\Gamma = m_\Gamma e r^{\alpha_\Gamma}$, r divides both $q^{m_\Gamma e r^{\alpha_\Gamma}} - \varepsilon^{m_\Gamma}$ and $q^{e_\Gamma\delta_\Gamma} - \varepsilon_\Gamma^{e_\Gamma}$, so $\varepsilon_\Gamma^{e_\Gamma} = \varepsilon^{m_\Gamma}$ and $R_\Gamma'^*$ is a Sylow r-subgroup of G_Γ^{0*}. In particular, $R_\Gamma'^*$ is cyclic of order $r^{a+\alpha_\Gamma}$ and has type $R_{m_\Gamma, \alpha_\Gamma, 0}$ as a subgroup of $I(V_\Gamma^*)$. Let R_Γ^* be the Sylow r-subgroup of T_Γ^*. Then R_Γ^* is cyclic of order $r^{a+\alpha_\Gamma}$ and there exists $g \in I(V_\Gamma^*)$ such that $(R_\Gamma^*)^g = R_\Gamma'^*$, so that $(C_\Gamma^*)^g = C_\Gamma'^*$. Thus $(T_\Gamma^*)^{gh} = T_\Gamma'^*$, and $s_\Gamma^{h^{-1}g^{-1}} \in T_\Gamma^*$ for some $h \in C_\Gamma'^*$. Thus $s_\Gamma^{h^{-1}g^{-1}}$ is a dual of s_Γ^* in G_Γ^{0*} and $C_{C_\Gamma^*}(s_\Gamma^{h^{-1}g^{-1}}) = T_\Gamma^*$. We may denote $s_\Gamma^{h^{-1}g^{-1}}$ by s_Γ and then the claim holds. By (3E) s_Γ is uniquely determined by Γ up to conjugacy in $I(V_\Gamma^*)$.

Let ϕ_Γ be the character of T_Γ corresponding to s_Γ, and let

$$\theta_\Gamma = \pm R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma) = \pm R_{T_\Gamma^*}^{C_\Gamma^*}(s_\Gamma),$$

where the sign is chosen so that θ_Γ is an irreducible character of C_Γ. The block b_Γ of C_Γ containing θ_Γ then has defect group R_Γ by [11, (4C)] and the Brauer pair (R_Γ, b_Γ) of G^0_Γ has the label $(R_\Gamma, s_\Gamma, -)$.

(3G). *Let* $N_\Gamma = N_{G_\Gamma}(R_\Gamma)$, *and* $N(\theta_\Gamma)$ *the stabilizer of* θ_Γ *in* N_Γ.

(a) $(N(\theta_\Gamma)\colon C_\Gamma) = \beta_\Gamma e_\Gamma$. *In particular,* $|\mathrm{Irr}^0(N(\theta_\Gamma), \theta_\Gamma)| = \beta_\Gamma e_\Gamma$ *and* R_Γ *is a defect group of* $b_\Gamma^{G_\Gamma}$.

(b) *Let* $\Gamma, \Gamma' \in \mathscr{F}'$ *such that* $G_\Gamma = G_{\Gamma'}$ *and* $R_\Gamma = R_{\Gamma'}$, *so that* $C_\Gamma = C_{\Gamma'}$ *and* $N_\Gamma = N_{\Gamma'}$. *Let* θ_Γ *and* $\theta_{\Gamma'}$ *be the canonical characters of* b_Γ *and* $b_{\Gamma'}$ *respectively. Then* $b_\Gamma^\tau = b_{\Gamma'}$ *for some* $\tau \in N_\Gamma$ *if and only if* s_Γ *and* $s_{\Gamma'}$ *are conjugate in* $I(V_\Gamma^*)$, *where* V_Γ^* *is the underlying space of* G_Γ^{0*}.

Proof. (a) It suffices to show $(N(\theta_\Gamma)\colon C_\Gamma) = \beta_\Gamma e_\Gamma$ since N_Γ/C_Γ is cyclic of order $2er^{\alpha_\Gamma}$. If $\Gamma \in \mathscr{F}_0$, then $C_\Gamma = T_\Gamma$, $\theta_\Gamma = \phi_\Gamma$, and θ_Γ is either the identity character or the character of order 2 of T_Γ. Thus $N(\theta_\Gamma) = N_\Gamma$ and $(N(\theta_\Gamma)\colon C_\Gamma) = 2e_\Gamma$.

Suppose $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, so that $T_\Gamma = C_{C_\Gamma}(\rho)$ for some $\rho \in T_\Gamma$ and $T_\Gamma = C_{G_\Gamma}(\mu\rho)$ for any generator μ of R_Γ. Let Δ be the unique elementary divisor of $\mu\rho$ and $N(T_\Gamma) = N_{G_\Gamma}(T_\Gamma)$. Following [12, p. 149], if $\Delta \in \mathscr{F}_1$, we have $N(T_\Gamma) = \langle \sigma, T_\Gamma\rangle$, where $\sigma\colon t \mapsto t^q$ for $t \in T_\Gamma$. Here σ has order $2m_\Gamma er^{\alpha_\Gamma}$ in $N(T_\Gamma)/T_\Gamma$ and $\sigma^{m_\Gamma er^{\alpha_\Gamma}}$ inverts T_Γ. If $\Delta \in \mathscr{F}_2$, we have $N(T_\Gamma) = \langle \beta, \gamma, T_\Gamma\rangle$, where $\beta\colon t \mapsto t^q$, $\gamma\colon t \mapsto t^{-1}$ for $t \in T_\Gamma$. Here β and γ have order $m_\Gamma er^{\alpha_\Gamma}$ and 2 respectively in $N(T_\Gamma)/T_\Gamma$. Moreover, $N_\Gamma = N(T_\Gamma)C_\Gamma$.

Let N_Γ act on the pairs (T, ϕ) by conjugation and let $[T, \phi]$ be the C_Γ-orbit of the pair (T, ϕ), where T is a Coxeter torus of C_Γ and ϕ is an irreducible character of T. Then N_Γ induces an action on the C_Γ-orbits and the N_Γ-orbit of $[T_\Gamma, \phi_\Gamma]$ consists of $\{[T_\Gamma, \phi_\Gamma^{\pm q^l}]\}$, where $1 \le l \le m_\Gamma er^{\alpha_\Gamma}$. Moreover, we claim that for $\tau \in N_\Gamma$, $\varrho \in N(T_\Gamma)$, $[T_\Gamma, \phi_\Gamma]^\tau = [T_\Gamma, \phi_\Gamma^\varrho]$ if and only if $(R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma))^\tau = R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma^\varrho)$. Indeed given $\tau \in N_\Gamma$, then $T_\Gamma^{\tau\varpi} = T_\Gamma$ for some $\varpi \in C_\Gamma$ and $(R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma))^\tau = (R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma))^{\tau\varpi} = R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma^{\tau\varpi})$. Thus $[T_\Gamma, \phi_\Gamma]^\tau = [T_\Gamma, \phi_\Gamma^\varrho]$ if and only if $[T_\Gamma, \phi_\Gamma^{\tau\varpi}] = [T_\Gamma, \phi_\Gamma^\varrho]$ if and only if $(R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma))^\tau = R_{T_\Gamma}^{C_\Gamma}(\phi_\Gamma^\varrho)$. Thus the claim holds. In particular, $N(\theta_\Gamma)$ is the stabilizer of $[T_\Gamma, \phi_\Gamma]$ in N_Γ.

The group C_Γ^* acts on the pairs (T^*, s) of Coxeter torus T^* and $s \in T^*$ by conjugation. Let $[T^*, s]$ be the conjugacy C_Γ^*-class of (T^*, s). By [18, (7.5)] the C_Γ-classes $[T, \phi]$ are in bijection with the C_Γ^*-classes $[T^*, s]$ and if $[T, \phi]$ corresponds to $[T^*, s]$, then $R_T^{C_\Gamma}(\phi) = R_{T^*}^{C_\Gamma^*}(s)$ and $[T, \phi^k]$ corresponds to $[T^*, s^k]$ for any integer k. Let R_Γ^* be the Sylow r-subgroup of T_Γ^* and $N_\Gamma^* = N_{I(V_\Gamma^*)}(R_\Gamma^*)$. Then $|R_\Gamma^*| = r^{a+\alpha_\Gamma}$, $R_\Gamma^* \le Z(C_\Gamma^*)$, and R_Γ^* has form $R_{m_\Gamma, \alpha_\Gamma, 0}$ as a subgroup of $I(V_\Gamma^*)$. So $C_\Gamma^* = C_{I_0(V_\Gamma^*)}(R_\Gamma^*)$. Let $N(T_\Gamma^*) = N_{I(V_\Gamma^*)}(T_\Gamma^*)$. Then $N_\Gamma^* = N(T_\Gamma^*)C_\Gamma^*$ and N_Γ^* acts on the pairs (T^*, s) by conjugation, so that N_Γ^* induces an action on classes $[T^*, s]$. If G_Γ is a symplectic group, then $I(V_\Gamma^*) \simeq \mathrm{O}(\beta_\Gamma e_\Gamma d_\Gamma + 1, q)$, and the action of $N(T_\Gamma^*)$ on T_Γ^* is similar to that of $N(T_\Gamma)$ on T_Γ, namely for $g \in N(T_\Gamma^*)$, g acts on T_Γ^* by $g\colon t \mapsto t^{\pm q^l}$, where $t \in T_\Gamma^*$ and $1 \le l \le m_\Gamma er^{\alpha_\Gamma}$. If G_Γ is an orthogonal group, then $I(V_\Gamma^*) \simeq I(V_\Gamma) \simeq \mathrm{O}^\pm(\beta_\Gamma e_\Gamma d_\Gamma, q)$ and the action of $N(T_\Gamma^*)$ on T_Γ^* is similar to that of $N(T_\Gamma)$ on T_Γ. Thus the N_Γ^*-orbit of $[T_\Gamma^*, s_\Gamma]$ consists of $\{[T_\Gamma^*, s_\Gamma^{\pm q^l}]\}$, where $1 \le l \le m_\Gamma er^{\alpha_\Gamma}$ and the elements in this orbit are in bijection with that in the N_Γ-orbit of $[T_\Gamma, \phi_\Gamma]$. So $(N_\Gamma\colon N(\theta_\Gamma)) = (N_\Gamma^*\colon N([T_\Gamma^*, s_\Gamma]))$, where $N([T_\Gamma^*, s_\Gamma])$

is the stabilizer of $[T_\Gamma^*, s_\Gamma]$ in N_Γ^*. Let $H^* = N([T_\Gamma^*, s_\Gamma])$ or $N([T_\Gamma^*, s_\Gamma]) \cap I_0(V_\Gamma^*)$ according as V_Γ is orthogonal or symplectic. Then $H^* \geq C_\Gamma^*$ and $|N(\theta_\Gamma)| = |H^*|$ since $|N_\Gamma| = |N_\Gamma^*|$ or $\frac{1}{2}|N_\Gamma^*|$ according as V_Γ is orthogonal or symplectic. Moreover, $(N(\theta_\Gamma): C_\Gamma) = (H^* : C_\Gamma^*)$.

Now fix the C_Γ^*-classes $[T_\Gamma^*, s_\Gamma]$. Then it is clear that C_Γ^* and H^* act transitively on the class and so $(H^*: N_{H^*}(T_\Gamma^*, s_\Gamma)) = (C_\Gamma^*: N_{C_\Gamma^*}(T_\Gamma^*, s_\Gamma))$, where $N_{H^*}(T_\Gamma^*, s_\Gamma)$ and $N_{C_\Gamma^*}(T_\Gamma^*, s_\Gamma)$ are the stabilizers of the pair (T_Γ^*, s_Γ) in H^* and C_Γ^* respectively. But $H^* \geq C_\Gamma^*$, $N_{C_\Gamma^*}(T_\Gamma^*, s_\Gamma) = T_\Gamma^*$, and

$$(H^* : T_\Gamma^*) = (H^* : C_\Gamma^*)(C_\Gamma^* : T_\Gamma^*) = (H^* : N_{H^*}(T_\Gamma^*, s_\Gamma))(N_{H^*}(T_\Gamma^*, s_\Gamma): T_\Gamma^*),$$

so $(H^* : C_\Gamma^*) = (N_{H^*}(T_\Gamma^*, s_\Gamma): T_\Gamma^*)$. If V_Γ is orthogonal, then $C_{I(V_\Gamma^*)}(s_\Gamma) = C_{I_0(V_\Gamma^*)}(s_\Gamma)$ by $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$. Thus in any case $N_{H^*}(T_\Gamma^*, s_\Gamma) \leq I_0(V_\Gamma^*)$. Let $K^* = C_{I_0(V_\Gamma^*)}(s_\Gamma)$. Then $K^* \simeq \mathrm{GL}(e_\Gamma, \varepsilon_\Gamma q^{\delta_\Gamma})$ and $N_{H^*}(T_\Gamma^*, s_\Gamma) = N_{K^*}(T_\Gamma^*)$. Since T_Γ^* is a Coxeter torus of K^*, $(N_{K^*}(T_\Gamma^*): T_\Gamma^*) = e_\Gamma$ and then $(N(\theta_\Gamma): C_\Gamma) = e_\Gamma$.

(b) Let $\theta_{\Gamma'} = \pm R_{T_{\Gamma'}^*}^{C_\Gamma^*}(s_{\Gamma'})$. Suppose $\theta_\Gamma^\tau = \theta_{\Gamma'}$ for some $\tau \in N_\Gamma$. Then $[T_\Gamma, \phi_\Gamma]^\tau$ corresponds to $[T_\Gamma^*, s_\Gamma^n]$ for some $n \in N(T_\Gamma^*)$ since the elements in the N_Γ-orbit of $[T_\Gamma, \phi_\Gamma]$ are in bijection with elements in the N_Γ^*-orbit of $[T_\Gamma^*, s_\Gamma]$ and $N_\Gamma^* = N(T_\Gamma^*)C_\Gamma^*$. Thus $\theta_\Gamma^\tau = \pm R_{T_\Gamma^*}^{C_\Gamma^*}(s_\Gamma^n)$ and $[T_\Gamma^*, s_\Gamma^n] = [T_{\Gamma'}^*, s_{\Gamma'}]$. So s_Γ is conjugate with $s_{\Gamma'}$ in $I(V^*)$. Conversely, suppose s_Γ and $s_{\Gamma'}$ are conjugate in $I(V_\Gamma^*)$. Since T_Γ^* and $T_{\Gamma'}^*$ are Coxeter tori of C_Γ^*, $T_{\Gamma'}^{*c} = T_\Gamma^*$ and $s_{\Gamma'}^c = s_\Gamma^w$ for some $c \in C_\Gamma^*$ and $w \in I(V_\Gamma^*)$. If $\Gamma \in \mathscr{F}_0$, then $C_\Gamma^* = T_\Gamma^* = T_{\Gamma'}^*$ and $s_{\Gamma'} = s_\Gamma^w$, so that both $s_{\Gamma'}$ and s_Γ are elements of T_Γ^* of order 1 or 2 according as $\Gamma = X - 1$ or $\Gamma = X + 1$. Thus $s_{\Gamma'} = s_\Gamma$ and $\theta_{\Gamma'} = \theta_\Gamma$. Suppose $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, so that $K^* = C_{I_0(V_\Gamma^*)}(s_{\Gamma'})^{cw^{-1}}$ and hence T_Γ^*, $T_{\Gamma'}^{*cw^{-1}}$ are Coxeter tori of K^*. So $T_\Gamma^{*g} = T_{\Gamma'}^{*cw^{-1}}$ for some $g \in K^*$, $T_\Gamma^{*gw} = T_{\Gamma'}^{*c} = T_\Gamma^*$, and $gw \in N_\Gamma^*$. It follows that

$$[T_\Gamma^*, s_\Gamma]^{gw} = [T_\Gamma^*, s_\Gamma^w] = [T_\Gamma^{*c^{-1}}, s_\Gamma^{wc^{-1}}] = [T_{\Gamma'}^*, s_{\Gamma'}].$$

Since $gw \in N_\Gamma^*$, $[T_\Gamma^*, s_\Gamma]^{gw}$ corresponds to $[T_\Gamma, \phi_\Gamma]^\tau$ for some $\tau \in N_\Gamma$ and then $\theta_{\Gamma'} = \theta_\Gamma^\tau$. This completes the proof.

Remark. Let G_Γ be an orthogonal group, and $N_0(\theta_\Gamma) = N(\theta_\Gamma) \cap G_\Gamma^0$. By [12, (6B)] $(N(\theta_\Gamma) : N_0(\theta_\Gamma)) = \beta_\Gamma$.

For each $\alpha \geq 0$ and $m \geq 0$, let $V_{m,\alpha,0}$ denote a symplectic or orthogonal space over $\mathbb{F}_q$ of dimension $2mer^\alpha$ and type ε^m if $V_{m,\alpha,0}$ is orthogonal. Thus $I(V_{m,\alpha,0})$ has a basic subgroup of form $R_{m,\alpha,0}$ (see §2).

(3H). *Let $G = I(V_{m,\alpha,0})$, $R = R_{m,\alpha,0}$ a basic subgroup of G, b a block of $C_G(R)R$ with defect group R, and θ the canonical character of b. If $N(\theta)$ is the stabilizer of θ in $N_G(R)$ and $(N(\theta): C_G(R)R)_r = 1$, then $G = G_\Gamma$, $R = R_\Gamma$, and $\theta = \theta_\Gamma$ for some $\Gamma \in \mathscr{F}'$.*

Proof. Let $C = C_G(R)$, $N = N_G(R)$, and $G_0 = I_0(V_{m,\alpha,0})$. Then $C = C_{G_0}(R)$ and N/C is cyclic of order $2er^\alpha$.

Since $C \simeq \mathrm{GL}(m, \varepsilon q^{er^\alpha})$, it follows by [11, (4B) and (4C)] that

$$\theta = \varepsilon_T R_T^C(\phi),$$

where $\varepsilon_T = \pm 1$, T is a Coxeter torus of C and ϕ is an r-rational irreducible character of T. Moreover, the dual T^* is embedded as a regular subgroup of

C^*, and C^* is embedded as a regular subgroup of G_0^*. There is an element s of T^* such that s corresponds to ϕ and $T^* = C_{C^*}(s)$. In particular, if $\phi^2 = 1$, then $s^2 = 1$, $T^* = C^*$, $m = 1$, and $\theta = \phi$. Thus $N = N(\theta)$ and $(N(\theta): C)_r = (N : C)_r = 1$, so that $\alpha = 0$. In this case $R = R_{X\pm 1}$, and $\theta = \theta_{X\pm 1}$ (see [12, p. 148]).

Suppose $\phi^2 \neq 1$. Then as an element of C^*, s has a unique elementary divisor Δ with multiplicity 1 since $T^* = C_{C^*}(s)$ is the Coxeter torus of C^*. Regard s as an element of G_0^*. By [12, (9A) and (9.2)] there is a unique $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ such that the multiplicity of Γ in s is $e_\Gamma r^l$ and $e_\Gamma r^l d_\Gamma = 2mer^\alpha$ for some $l \geq 0$. So $C_{G_0^*}(s) \simeq \mathrm{GL}(e_\Gamma r^l, \varepsilon_\Gamma q^{\delta_\Gamma})$. A similar proof to that of (3G)(a) shows that $(N(\theta): C) = (N_{C_{G_0^*}(s)}(T^*): T^*) = e_\Gamma r^l$. Thus $l = 0$ and $e_\Gamma d_\Gamma = 2mer^\alpha$ since $(N(\theta): C)_r = 1$. But $(m, r) = 1$ by [11, (4B)]. It follows that $m = m_\Gamma$, $\alpha = \alpha_\Gamma$, and $G = G_\Gamma$, $R = R_\Gamma$, $\theta = \theta_\Gamma$. This completes the proof.

Given $\Gamma \in \mathscr{F}'$ and $\gamma \geq 0$. Let

$$V_{\Gamma,\gamma} = V_\Gamma \perp V_\Gamma \perp \cdots \perp V_\Gamma, \tag{3.12}$$

where there are r^γ terms V_Γ on the right-hand side. Then if V_Γ is orthogonal, $V_{\Gamma,\gamma}$ has type $(\varepsilon_\Gamma)^{e_\Gamma r^\gamma} = \varepsilon_\Gamma^{e_\Gamma}$ or $\varepsilon^{r^\gamma} = \varepsilon$ according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$.

(3I). *Let* $G = I(V_{\Gamma,\gamma})$, $R = R_{m_\Gamma, \alpha_\Gamma, \gamma}$ *a basic subgroup of* G, *and* $C = C_G(R)$. *Then* $C = C_\Gamma \otimes I_\gamma$, *where* I_γ *is the identity matrix of order* r^γ. *The irreducible character* $\theta = \theta_\Gamma \otimes I_\gamma$, *of* C *defined by* $\theta(c \otimes I_\gamma) = \theta_\Gamma(c)$ *for* $c \in C_\Gamma$ *is then a character of defect* 0 *of* CR/R, *and* $|\mathrm{Irr}^0(N(\theta), \theta)| = \beta_\Gamma e_\Gamma$.

Proof. The proof is essentially the same as that of (3A), except that the automorphisms on $C = C_\Gamma \otimes I_\gamma$ induced by $N(R)$ have order $2er^{\alpha_\Gamma}$, and their actions are the same as the automorphisms on C_Γ induced by N_Γ/C_Γ.

Remark. Suppose $G = I(V_{\Gamma,\gamma})$ is an orthogonal group. Let $G_0 = I_0(V_{\Gamma,\gamma})$ and $N_0(\theta) = N(\theta) \cap G_0$. Then $|N(\theta): N_0(\theta)| = \beta_\Gamma$ and for each $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$ the restriction $\psi|_{N_0(\theta)}$ of ψ to $N_0(\theta)$ is irreducible. Indeed let $N^0 = \{g \in N: [g, Z(R)] = 1\}$. Then $N^0 \leq N_0(\theta)$ and in the notation of (3A), $N(\theta) = N(\vartheta)$ and $N(\vartheta)/N^0 \simeq N(\theta_\Gamma)/C_\Gamma$, where ϑ is the unique irreducible character of N^0 covering θ and having defect 0 as a character of N^0/R. The remark of (3G) implies $|N(\theta): N_0(\theta)| = \beta_\Gamma$. Since ψ covers ϑ and $N(\vartheta)/N^0$ is cyclic, $\psi|_{N^0} = \vartheta$ is irreducible, so that $\psi|_{N_0(\theta)}$ is irreducible. This completes the proof.

Given $\Gamma \in \mathscr{F}'$, and $d \geq 0$. Let $G = I(V_{\Gamma,d})$, and $R = R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$ be a basic subgroup of G, where $\mathbf{c} = (c_1, c_2, \ldots, c_l)$, and $\gamma + c_1 + c_2 + \cdots + c_l = d$. Then $C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_\mathbf{c}$, where I_γ and $I_\mathbf{c}$ are the identity matrices of order r^γ and $r^{c_1+c_2+\cdots+c_l}$ respectively. The irreducible character of C defined by

$$\theta(c \otimes I_\gamma \otimes I_\mathbf{c}) = \theta_\Gamma(c) \tag{3.13}$$

for $c \in C_\Gamma$ is a character of defect 0 of CR/R. We shall say that the pair (R, θ) is of *type* Γ. If (R, θ) is of type Γ, then θ is a canonical character of a block b of C with defect group $Z(R)$, and the Brauer pair (R, b) of G is also a Brauer pair of $G_0 = I_0(V_{\Gamma,d})$ since $C = C_{G_0}(R)$. Let D be the base subgroup of $R = R_{m_\Gamma, \alpha_\Gamma, \gamma} \wr A_{\mathbf{c}}$. Then each component Q of D is of the form $R_{m_\Gamma, \alpha_\Gamma, \gamma}$, so that by the remark of (1C) Q contains a normal subgroup Q' such that $C_{I_0(V_{m_\Gamma, \alpha_\Gamma, \gamma})}(Q') = C_{I(V_{m_\Gamma, \alpha_\Gamma, \gamma})}(Q') = \prod_{i=1}^{r^\gamma} C_i$ is a regular subgroup of $I_0(V_{m_\Gamma, \alpha_\Gamma, \gamma})$, where $V_{m_\Gamma, \alpha_\Gamma, \gamma}$ is the underlying space of Q and $C_i \simeq \mathrm{GL}(m_\Gamma, \varepsilon q^{er^{\alpha_\Gamma}})$ for all i. Let R' be the subgroup of D with each component Q of D replaced by Q'. Then R' is a normal subgroup of R and $C' = C_G(R') = \prod_{i=1}^{r^d} C_i$, where $C_i \simeq \mathrm{GL}(m_\Gamma, \varepsilon q^{er^{\alpha_\Gamma}})$ for all $1 \leq i \leq r^d$. Thus C' is a regular subgroup of $I_0(V_{\Gamma, d})$ and $C \leq C'$, so that C'^* is embedded as a regular subgroup of $I_0(V_{\Gamma, d})^*$. Now we may suppose $C_i = C_\Gamma$ and $s_\Gamma \in C_i^*$ for all i. Let

$$x_\Gamma = s_\Gamma \times s_\Gamma \times \cdots \times s_\Gamma \qquad (r^d \text{ times}) \tag{3.14}$$

be an element of C'^* and x_Γ^* a dual of x_Γ in G. Then as an element of G, x_Γ^* has a unique elementary divisor Γ of multiplicity $\beta_\Gamma e_\Gamma r^d$ and type $\eta_\Gamma(x_\Gamma^*) = \eta(V_{\Gamma, d})$. The subgroup $C_\Gamma^* \otimes I_\gamma \otimes I_{\mathbf{c}}$ can be regarded as a diagonal subgroup of C'^*, so that $s_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}} \in C'^*$ and x_Γ is conjugate with $s_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$ in $I(V^*)$. Thus (R, b) is labeled by $(R, x_\Gamma, -)$. The Brauer pair (R, b) of G will also be denoted by (R, θ).

(3J). (a) *Let $G = I(V)$, R a basic subgroup of G, (R, φ) a weight of G, and θ an irreducible character of $C_G(R)$ covered by φ. Then (R, θ) is of type Γ for some $\Gamma \in \mathscr{F}'$.*

(b) *The pair (R, θ) of G with type Γ is uniquely determined by Γ up to conjugacy in $N = N_G(R)$, that is, if (R, θ') is another pair with type Γ, then $\theta' = \theta^n$ for some $n \in N$.*

Proof. (a) Suppose $V = V_{m, \alpha, \gamma, \mathbf{c}}$ and $R = R_{m, \alpha, \gamma, \mathbf{c}}$, where $\mathbf{c} = (c_1, \ldots, c_l)$. Let $G_1 = I(V_{m, \alpha, 0})$, $R_1 = R_{m, \alpha, 0}$ a basic subgroup of G_1, $C_1 = C_{G_1}(R_1)$, and $N_1 = N_{G_1}(R_1)$. Then $C_1 \simeq \mathrm{GL}(m, \varepsilon q^{e r^\alpha})$ and $C = C_G(R) = C_1 \otimes I_\gamma \otimes I_{\mathbf{c}}$. Thus θ has the form $\theta_1 \otimes I_\gamma \otimes I_{\mathbf{c}}$, where θ_1 is a character of C_1. Since θ has defect 0 as a character of $C/Z(R)$, θ_1 has defect 0 as a character on C_1/R_1. The block of C_1 containing θ_1 has defect group R_1.

Let $R_{m, \alpha, \gamma}$ a basic subgroup of $I(V_{m, \alpha, \gamma})$, $N_{m, \alpha, \gamma}$ and $C_{m, \alpha, \gamma}$ the normalizer and centralizer of $R_{m, \alpha, \gamma}$ in $I(V_{m, \alpha, \gamma})$. Then $C_{m, \alpha, \gamma} = C_1 \otimes I_\gamma$ and $(\theta_1 \otimes I_\gamma)(c \otimes I_\gamma) = \theta_1(c)$ for $c \in C_1$ is an irreducible character of $C_{m, \alpha, \gamma}$. By (2.5)

$$N = (N_{m, \alpha, \gamma}/R_{m, \alpha, \gamma}) \otimes N_{\mathbf{S}(u)}(A_{\mathbf{c}}),$$
$$N/R \simeq (N_{m, \alpha, \gamma}/R_{m, \alpha, \gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r),$$

where $u = r^{c_1 + \cdots + c_l}$. If $N^0_{m, \alpha, \gamma} = \{g \in N_{m, \alpha, \gamma} \colon [g, Z(R_{m, \alpha, \gamma})] = 1\}$, then $N_{m, \alpha, \gamma}/N^0_{m, \alpha, \gamma} \simeq N_1/C_1$. Let $\varphi = I(\psi)$ for some $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$, and $N(\theta_1 \otimes I_\gamma)$ be the stabilizer of $\theta_1 \otimes I_\gamma$ in $N_{m, \alpha, \gamma}$. Then

$$N(\theta)/R \simeq (N(\theta_1 \otimes I_\gamma)/R_{m, \alpha, \gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r).$$

But ψ is a character of defect 0 of $N(\theta)/R$, so it covers an irreducible character ψ_0 in $\mathrm{Irr}^0(N(\theta_1 \otimes I_\gamma), \theta_1 \otimes I_\gamma)$. Same proof as that of (3A) shows that $N^0_{m, \alpha, \gamma} \leq N(\theta_1 \otimes I_\gamma)$ and $N^0_{m, \alpha, \gamma}$ has a unique irreducible character ϑ covering $\theta_1 \otimes I_\gamma$ and having defect 0 as a character of $N^0_{m, \alpha, \gamma}/R_{m, \alpha, \gamma}$. Moreover, $N(\theta_1 \otimes I_\gamma) =$

$N(\vartheta)$ and $N(\vartheta)/N^0_{m,\alpha,\gamma} \simeq N(\theta_1)/C_1$, where $N(\theta_1)$ is the stabilizer of θ_1 in N_1. Thus $\psi_0 \in \operatorname{Irr}^0(N(\vartheta), \vartheta)$ and $\psi_0(1) = \vartheta(1)$ since $N_{m,\alpha,\gamma}/N^0_{m,\alpha,\gamma}$ is cyclic. By (3.1) $(N(\vartheta)\colon N^0_{m,\alpha,\gamma})_r = 1$ and hence $(N(\theta_1)\colon C_1)_r = 1$. It follows by (3H) that $G_1 = G_\Gamma$, $R_1 = R_\Gamma$, and $\theta_1 = \theta_\Gamma$ for some $\Gamma \in \mathscr{F}'$. Thus (R_1, θ_1) is labeled by $(R_1, s_\Gamma, -)$ and (R, θ) has type Γ, so (a) holds.

(b) Let $G = I(V_{\Gamma,d})$, $R = R_{m_\Gamma,\alpha_\Gamma,\gamma,\mathbf{c}}$ a basic subgroup of G, $C = C_G(R)$, $N = N_G(R)$, $\theta = \theta_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, and $\theta' = \theta'_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, where $\theta_\Gamma, \theta'_\Gamma$ are irreducible characters of C_Γ, and θ, θ' are defined as (3.13). If $(R_\Gamma, t_\Gamma, -)$ and $(R_\Gamma, t'_\Gamma, -)$ are the labels of $(R_\Gamma, \theta_\Gamma)$ and $(R_\Gamma, \theta'_\Gamma)$ respectively, then t_Γ, t'_Γ are conjugate in G_Γ since both Brauer pairs (R, θ) and (R, θ') are labeled by $(R, x_\Gamma, -)$. It follows by (3G)(b) that $\theta_\Gamma^w = \theta'_\Gamma$ for some $w \in N_\Gamma$.

Let $C_{m_\Gamma,\alpha_\Gamma,\gamma} = C_{I(V_{\Gamma,\gamma})}(R_{m_\Gamma,\alpha_\Gamma,\gamma})$, so that $C_{m_\Gamma,\alpha_\Gamma,\gamma} = C_\Gamma \otimes I_\gamma$. Let $\theta_\Gamma \otimes I_\gamma$ and $\theta'_\Gamma \otimes I_\gamma$ be irreducible characters of $C_{m_\Gamma,\alpha_\Gamma,\gamma}$ defined as (3I). Since $N_{m_\Gamma,\alpha_\Gamma,\gamma}/N^0_{m_\Gamma,\alpha_\Gamma,\gamma} \simeq N_\Gamma/C_\Gamma$, it follows $(\theta \otimes I_\gamma)^h = \theta'_\Gamma \otimes I_\gamma$ for some $h \in N_{m_\Gamma,\alpha_\Gamma,\gamma}$ and so $\theta^n = \theta'$ for some $n \in N$, where the structure of N is given above with m and α replaced by m_Γ and α_Γ respectively.

Remark. Suppose R is a basic subgroup of $G = I(V)$, b a block of $C_G(R)R$ with defect group R, and θ the canonical character of b. If $(N(\theta)\colon C_G(R)R)_r = 1$, then (R, θ) is of type Γ for some $\Gamma \in \mathscr{F}'$. In particular, this occurs when b is a root block of a block B and R is a defect group of B. Here a root block b of a block B, in the sense of Brauer, is a block of $C_G(R)R$ with defect group R such that $b^G = B$, where R is a defect group of B. Thus if b is a root block of B and θ is the canonical character of b, then (R, b) is a maximal Brauer pair containing $(1, B)$ and $(N(\theta)\colon C_G(R)R)_r = 1$, where b is regarded as a block of $C_G(R)$. The proof of the remark is similar to that of (3J)(a). Indeed in the notation of (3J)(a) $N(\theta_1 \otimes I_\gamma)/N^0_{m,\alpha,\gamma} \simeq N(\theta_1)/C_1$ and

$$N(\theta)/C_G(R)R \simeq (N(\theta_1 \otimes I_\gamma)/C_{m,\alpha,\gamma}R_{m,\alpha,\gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r).$$

Thus $(N(\theta_1 \otimes I_\gamma)\colon C_{m,\alpha,\gamma}R_{m,\alpha,\gamma})_r = 1$ and $(N(\theta_1 \otimes I_\gamma)\colon N^0_{m,\alpha,\gamma})_r = 1$ since $(N(\theta)\colon C(R)R)_r = 1$. So $(N(\theta_1)\colon C_1)_r = 1$ and the block of C_1 containing θ_1 has defect group R_1. By (3H) $G_1 = G_\Gamma$, $R_1 = R_\Gamma$, $\theta_1 = \theta_\Gamma$, and (R, θ) has type Γ.

Following the remark above we can get a corollary.

(3K). *Let V be a symplectic or even dimensional orthogonal space, $G = I(V)$, $G_0 = I_0(V)$, and let B and B' be blocks of G with defect D and D' respectively such that $[V, D] = V = [V, D']$. Let b and b' be root blocks of B and B' respectively, $b^{G_0} \subseteq \mathscr{E}_r(G_0, (s))$, and $b'^{G_0} \subseteq \mathscr{E}_r(G_0, (s'))$, where s and s' are semisimple r'-elements of G_0^*. Then $B = B'$ if and only if s and s' are conjugate in $I(V^*)$, where V^* is the underlying space of G_0^*.*

Proof. Since D is radical in G, a primary element of D exists and then G has an r-subgroup of form $R_{m,0,0}$ for some $m \geq 1$. By [12, (1.12) and (5.2)], V has dimension $2em$ and type ε^m if V is orthogonal.

Suppose s and s' are conjugate in $I(V^*)$, so that s^* and s'^* are conjugate in G by definition. By (3F) D and D' are conjugate with Sylow r-subgroups of $C_G(s^*)$ and $C_G(s'^*)$ respectively, so that they are conjugate in G. We may suppose $D = D'$.

By (2D) V and D have a corresponding decomposition,

$$V = V_1 \perp V_2 \perp \cdots \perp V_t, \qquad D = D_1 \times D_2 \times \cdots \times D_t,$$

where D_i is a basic subgroup of $I(V_i)$. Let θ and θ' be the canonical characters of b and b' respectively. Thus $C = C_G(D) = \prod_i C_i$, $\theta = \prod_i \theta_i$, and $\theta' = \prod_i \theta'_i$, where θ, θ' are regarded as characters of C, and θ_i, θ'_i are characters of $C_i = C_{I(V_i)}(D_i) = C_{I_0(V_i)}(D_i)$. Since b and b' are root blocks, it follows $(N(\theta) : CD)_r = (N(\theta') : CD)_r = 1$. Let $N(\theta_i)$ and $N(\theta'_i)$ be the stabilizers of θ_i and θ'_i in $N_{I(V_i)}(D_i)$ respectively. Then $(N(\theta_i) : C_iD_i)_r = (N(\theta'_i) : C_iD_i)_r = 1$ for all i. By the remark above, (D_i, θ_i) and (D_i, θ'_i) are of type Γ and Γ' respectively, where $\Gamma, \Gamma' \in \mathcal{F}'$.

Suppose $(D_i, t_i, -)$ and $(D_i, t'_i, -)$ are labels of Brauer pairs (D_i, θ_i) and (D_i, θ'_i) of $I(V_i)$ respectively. Let $z \in Z(D)$ be primary. Then we may suppose t_i and t'_i are elements of K_i^*, where $K_i = C_{I(V_i)}(z_i)$ and z_i is the restriction of z to V_i. So $(D, \prod_i t_i, -)$ is the label of (D, b) and $(D, \prod_i t'_i, -)$ is the label of (D, b'). By (3C), s and $\prod_i t_i$ are conjugate in G_0^*, so are s' and $\prod_i t'_i$. Thus these three elements s, $\prod_i t_i$, and $\prod_i t'_i$ are conjugate in $I(V^*)$. Let $D(\Gamma) = \prod_i D_i$, $s(\Gamma) = \prod_i t_i$, $V(\Gamma) = \sum_i V_i$, $D'(\Gamma') = \prod_j D_j$, and $s'(\Gamma') = \prod_j t'_j$, where i and j runs over indices such that (D_i, θ_i) and (D_j, θ'_j) have type Γ and Γ' respectively. Then $\prod_\Gamma s(\Gamma)$, $\prod_{\Gamma'} s(\Gamma')$, and s are conjugate in $I(V^*)$. Let z_Γ be the restriction of z to $V(\Gamma)$ and $K_\Gamma = C_{I(V(\Gamma))}(z_\Gamma)$. Then K_Γ^* can be embedded as a subgroup of $I_0(V(\Gamma))^*$ and $s(\Gamma) \in K_\Gamma^*$. If $s(\Gamma)^*$ is a dual of $s(\Gamma)$ in $I_0(V(\Gamma))$, then $\prod_\Gamma s(\Gamma)^*$ is a primary decomposition of s^*. Similarly, $\prod_\Gamma s'(\Gamma)^*$ is a primary decomposition of s^*. So $D(\Gamma)$ is a Sylow r-subgroup of $H_\Gamma = C_{I(V(\Gamma))}(s(\Gamma)^*)$, and $D(\Gamma)$, $D'(\Gamma)$ are conjugate in G.

If $\Gamma = X \pm 1$, then $\dim V(\Gamma) = 2ew_\Gamma$ for some $w_\Gamma \geq 0$ and $D(\Gamma) = \prod_\beta (R_{1,0,0,\mathbf{c}_\beta})^{n_\beta}$, where $n_\beta \geq 0$ such that $w_\Gamma = \sum_\beta n_\beta r^\beta$ is the r-adic expansion of w_Γ, and $\mathbf{c}_\beta = (1, 1, \ldots, 1)$ (β terms). If $\Gamma \neq X \pm 1$, then $H_\Gamma \simeq \mathrm{GL}(e_\Gamma w_\Gamma, \varepsilon_\Gamma q^{\delta_\Gamma})$ and $D(\Gamma) = \prod_\beta (R_{m_\Gamma, \alpha_\Gamma, 0, \mathbf{c}_\beta})^{n_\beta}$, where n_β and $\mathbf{c}_\beta$ defined as before.

Fix $1 \leq i \leq t$. If (D_i, θ_i) is of type Γ, then D_i is a component of $D(\Gamma)$, and so $D_i = R_{m_\Gamma, \alpha_\Gamma, 0, \mathbf{c}_\beta}$. Suppose for the same i, (D_i, θ'_i) is of type Γ'. Thus D_i is also a component of $D'(\Gamma')$, and hence $D_i = R_{m_{\Gamma'}, \alpha_{\Gamma'}, 0, \mathbf{c}_{\beta'}}$. So $m_\Gamma = m_{\Gamma'}$, $\alpha_\Gamma = \alpha_{\Gamma'}$, and $\beta = \beta'$. Since $D(\Gamma')$ and $D'(\Gamma')$ are conjugate in G, there exists a component D_j of $D(\Gamma')$, for $1 \leq j \leq t$, such that (D_j, θ_j) is of type Γ' and $D_j = R_{m_{\Gamma'}, \alpha_{\Gamma'}, 0, \mathbf{c}_{\beta'}}$. So D_j and D_i have the same form $R_{m_\Gamma, \alpha_\Gamma, 0, \mathbf{c}_\beta}$. By (2E), there exists $h \in N_G(D)$ permuting (V_i, D_i), (V_j, D_j) and holding (V_k, D_k) fixed for $k \neq i, j$. Thus h permutes (D_i, θ_i), (D_j, θ_j) and holds (D_k, θ_k) fixed for $k \neq i, j$. Replacing (D_i, θ_i) by $(D_i, \theta_i)^h$, we may suppose both (D_i, θ_i) and (D_i, θ'_i) are of the same type Γ', and we may suppose this for all $i \geq 1$. By (3J)(b), for each i, $\theta_i^{g_i} = \theta'_i$ for some $g_i \in N_{I(V_i)}(D_i)$ and then $\theta^g = \theta'$ for some $g \in N_G(D)$. It follows that $B = b^G = b'^G = B'$.

Conversely, suppose $B = B'$. If $b^{G_0} = b'^{G_0}$, which occurs when G is a symplectic group, then s and s' are conjugate in G_0^* by (3C).

Suppose $b^{G_0} \neq b'^{G_0}$. Then G is an orthogonal group, and $(b^{G_0})^g = b'^{G_0}$ for some $g \in G$ of determinant -1. So B covers exactly two blocks b^{G_0} and b'^{G_0} of G_0. Let $N_0(\theta)$ be the stabilizer of θ in $N_{G_0}(D)$. Then $(N(\theta) : N_0(\theta)) = 1$

or 2. If $(N(\theta) : N_0(\theta)) = 2$, then $\theta^x = \theta$ for some $x \in G$ of determinant -1. So $(b^{G_0})^x = b^{G_0}$ and thus $(b^{G_0})^g = b^{G_0}$ for all $g \in G$. This is impossible. Thus $N(\theta) = N_0(\theta)$ and then $m_{X\pm 1}(s) = 0$ by [12, (7B) and (7C)]. It follows that $C_{I(V^*)}(s) = C_{G_0^*}(s)$, so there exists $x \in I(V^*)$ of determinant -1 such that s^x and s are not conjugate in G_0^*. Let D_x be a Sylow r-subgroup of $C_{G_0}(s^{x*})$, and $y^* \in Z(D_x)$ primary. Thus D_x and D are conjugate in G, and $s^{x*} \in C_G(y^*) \simeq \mathrm{GL}(m, \varepsilon q^e)$ for some $m \geq 1$. Let y be a primary element of a Sylow r-subgroup of $Z(C_G(y^*)^*)$. Then $C_{G_0^*}(y) = C_G(y^*)^*$ and $\langle y \rangle$ is conjugate in $I(V^*)$ with the subgroup generated by a dual of y^*, so y^k is a dual of y^* for some integer $k \geq 1$ and $\langle y \rangle = \langle y^k \rangle$ by $|y^k| = r^a$. By the remark of (3E) we may suppose s^x lies in the r-section containing y^k and $s^x \in C_G(y^*)^*$. There exists a block b_x of $C_{G_0}(y^*)$ labeled by $(s^x, -)$, so that $(\langle y^* \rangle, b_x)$ is a Brauer pair of G_0 labeled by $(\langle y^* \rangle, s^x, -)$ and $b_x^{G_0} \subseteq \mathscr{E}_r(G_0, (s^x))$ by (3C). Since s and s^x are conjugate in $I(V^*)$, it follows that $b^G = B = b_x^G$ by the first half of the proof and so B covers $b_x^{G_0}$. Since s and s^x are not conjugate in G_0^*, it follows that $b^{G_0} \neq b_x^{G_0}$ and so $b_x^{G_0} = b'^{G_0}$. Thus s^x and s' are conjugate in G_0^*. This completes the proof.

4. Weights for classical groups

In this section we count the number of B-weights for a block B of finite classical groups. Given $\Gamma \in \mathscr{F}'$ and integer $d \geq 0$, let $V_{\Gamma, d}$ be a unitary space of dimension $r^d e_\Gamma d_\Gamma$ over $\mathbb{F}_{q^2}$, or a symplectic or orthogonal space over $\mathbb{F}_q$ given by (3.12). Denote $G = G_0 = \mathrm{U}(V_{\Gamma, d})$ in the case $V_{\Gamma, d}$ is unitary, and $G = I(V_{\Gamma, d})$, $G_0 = I_0(V_{\Gamma, d})$ in the remaining cases. Let $0 \leq \gamma \leq d$, and $\mathbf{c} = (c_1, c_2, \dots, c_l)$ a sequence of nonnegative integers such that $d - \gamma = c_1 + c_2 + \cdots + c_l$. In addition, let

$$R = R_{m_\Gamma, \alpha_\Gamma, \gamma} \wr A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_l},$$

be a basic subgroup of G, $C = C_G(R)$, and $N = N_G(R)$. Then $C = C_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, where I_γ and $I_{\mathbf{c}}$ are identity matrices of orders r^γ and $r^{c_1 + c_2 + \cdots + c_l}$ respectively. Define θ on C by $\theta(c \otimes I_\gamma \otimes I_{\mathbf{c}}) = \theta_\Gamma(c)$ for $c \in C_\Gamma$. Then θ is an irreducible character of C and (R, θ) is of type Γ. Regard θ as a character of CR trivial on R. Then the block b of CR containing θ has defect group R and the Brauer pair (R, b) of G_0 has label $(R, x_\Gamma, -)$, where b is regarded as a block of C, and $x_\Gamma = r^d e_\Gamma \Gamma$ in the case G is unitary and x_Γ is given by (3.14) in the remaining cases. Let $V_{m_\Gamma, \alpha_\Gamma, \gamma}$ be the underlying space of $R_{m_\Gamma, \alpha_\Gamma, \gamma}$, $G_{m_\Gamma, \alpha_\Gamma, \gamma} = \mathrm{U}(V_{m_\Gamma, \alpha_\Gamma, \gamma})$ in the case $V_{m_\Gamma, \alpha_\Gamma, \gamma}$ is unitary, or $I(V_{m_\Gamma, \alpha_\Gamma, \gamma})$ in the remaining case. If $\theta_\Gamma \otimes I_\gamma$ is the character of $C_{G_{m_\Gamma, \alpha_\Gamma, \gamma}}(R_{m_\Gamma, \alpha_\Gamma, \gamma}) = C_\Gamma \otimes I_\gamma$ defined by $(\theta_\Gamma \otimes I_\gamma)(c \otimes I_\gamma) = \theta_\Gamma(c)$ for $c \in C_\Gamma$ and $N(\theta_\Gamma \otimes I_\gamma)$ is its stabilizer in $N_{G_{m_\Gamma, \alpha_\Gamma, \gamma}}(R_{m_\Gamma, \alpha_\Gamma, \gamma})$, then by (2.2) or (2.5)

$$\begin{aligned} N(\theta) &= (N(\theta_\Gamma \otimes I_\gamma)/R_{m_\Gamma, \alpha_\Gamma, \gamma}) \otimes N_{\mathrm{S}(u)}(A_{\mathbf{c}}), \\ N(\theta)/R &\simeq (N(\theta_\Gamma \otimes I_\gamma)/R_{m_\Gamma, \alpha_\Gamma, \gamma}) \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_l, r). \end{aligned} \tag{4.1}$$

Thus the characters ψ in $\mathrm{Irr}^0(N(\theta), \theta)$ are parametrized by $(l+1)$-tuples $(\psi_0, \psi_1, \dots, \psi_l)$, where $\psi_0 \in \mathrm{Irr}^0(N(\theta_\Gamma \otimes I_\gamma), \theta_\Gamma \otimes I_\gamma)$ and ψ_i is an irreducible character of $\mathrm{GL}(c_i, r)$ of defect 0 for $i \geq 1$. Necessarily, ψ_i are one of the $r-1$ Steinberg characters of $\mathrm{GL}(c_i, r)$ for $i \geq 1$. By (3A) or (3I) there are

$\beta_\Gamma e_\Gamma$ such characters ψ_0, so that there are $\beta_\Gamma e_\Gamma (r-1)^l$ such characters ψ, where $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$. Thus there are $\beta_\Gamma e_\Gamma (r-1)^l$ b^G-weights of the form $(R, I(\psi))$.

(4A). *Let $V = V_{\Gamma, d}$, B a block of G with defect group D and root block* $\mathbf{b}$ *such that $[V, D] = V$ and $\mathbf{b}^{G_0} \subseteq \mathscr{E}_r(G_0, (x_\Gamma))$. Then there are exactly $\beta_\Gamma e_\Gamma r^d$ B-weights (R, φ), where R runs over the basic subgroups of G with degree $\beta_\Gamma e_\Gamma d_\Gamma r^d$.*

Proof. (1) Suppose $R = R_{m,\alpha,\gamma,\mathbf{c}}$ is a basic subgroup of G, (R, φ) is a B-weight, and φ covers the irreducible character θ of $C_G(R)R$. Then the block b of $C_G(R)R$ containing θ has a defect group R and $b^G = B$. By (3B) or (3J)(a) (R, θ) has type Δ for some $\Delta \in \mathscr{F}'$ and (R, b) has label $(R, x_\Delta, -)$, where b is regarded as a block of $C_G(R)$. If V is unitary, then $\Delta = \Gamma$ by [7, (3.2)]. Suppose V is a symplectic or orthogonal space. Let $(D', \mathbf{b}')$ be a maximal pair containing (R, b), so that $\mathbf{b}'^G = B$. As a block of $C_G(D')D'$, $\mathbf{b}'$ is also a root block of B and $b^{G_0} = \mathbf{b}'^{G_0} \subseteq \mathscr{E}_r(G_0, (x_\Delta))$ by (3C). Since D' is a defect group of B, D' and D are conjugate in G and so $[V, D'] = V$. By (3K) x_Δ and x_Γ are conjugate in $I(V^*)$, where V^* is the underlying space of G_0^*. Thus $\Delta = \Gamma$ and $m = m_\Gamma$, $\alpha = \alpha_\Gamma$, $\gamma + c_1 + c_2 + \cdots + c_l = d$.

The number of different sequences $\mathbf{c} = (c_1, c_2, \ldots, c_l)$ such that

$$d(R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}) = \beta_\Gamma e_\Gamma d_\Gamma r^d \quad \text{and} \quad l(R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}) = l$$

is $\binom{d-\gamma-1}{l-1}$. Here $1 \le l \le d - \gamma$ when $d - \gamma \ge 1$; $l = 0$ when $d = \gamma$, and $\binom{-1}{-1}$ is interpreted as 1. There are $\beta_\Gamma e_\Gamma (r-1)^l$ characters φ associated with $R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$, so that there are

$$\beta_\Gamma e_\Gamma \sum_{\gamma=0}^{d} \sum_{l=0}^{d-\gamma} \binom{d-\gamma-1}{l-1} (r-1)^l = \beta_\Gamma e_\Gamma r^d,$$

characters associated with $R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$'s.

(2) Suppose V is a symplectic or orthogonal space. By (3J)(b) the pair (R, θ) of type Γ is determined uniquely up to conjugacy in $N_G(R)$, so that there are $\beta_\Gamma e_\Gamma r^d$ B-weights (R, φ). Suppose V is a unitary space and (R, b') is another Brauer pair of G such that $b'^G = B$, and θ' is the canonical character of b', where $R = R_{m_\Gamma, \alpha_\Gamma, \gamma, \mathbf{c}}$. Then (R, θ') has type Γ, $C = C_G(R) = C_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, and θ' has the form $\theta'_\Gamma \otimes I_\gamma \otimes I_{\mathbf{c}}$, where θ'_Γ is an irreducible character of C_Γ. If b'_Γ is the block of C_Γ containing θ'_Γ, then $b_\Gamma'^{G_\Gamma} = B_\Gamma$ and both B_Γ and b'_Γ have a defect group R_Γ. By definition $b_\Gamma^{G_\Gamma} = B_\Gamma$ and b_Γ has a defect group R_Γ. Thus $b_\Gamma^w = b'_\Gamma$ for some $w \in N_\Gamma$ by Brauer First Main Theorem. A similar proof to that of (3J)(b) shows that $\theta' = \theta^n$ for some $n \in N_G(R)$. Thus (4A) follows in this case.

Remark. In the notation of (4A), suppose V is orthogonal, $G = I(V)$, and $G_0 = I_0(V)$. If (R, θ) has type Γ, then $|N(\theta) : N_0(\theta)| = \beta_\Gamma$ and for each $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$, the restriction $\psi|_{N_0(\theta)}$ of ψ to $N_0(\theta)$ is irreducible, where $N_0(\theta) = N(\theta) \cap G_0$. Indeed in the notation above $\psi = (\psi_0, \psi_1, \ldots, \psi_l)$ as a character of $N(\theta)/R$, where $\psi_0 \in \operatorname{Irr}^0(N(\theta_\Gamma \otimes I_\gamma), \theta_\Gamma \otimes I_\gamma)$, and ψ_i is an irreducible character of $\mathrm{GL}(c_i, r)$ of defect 0 for $i \ge 1$. Let $N_0(\theta_\Gamma \otimes I_\gamma)$ be the subgroup of $N(\theta_\Gamma \otimes I_\gamma)$ of determinant 1. Then $|N(\theta_\Gamma \otimes I_\gamma) : N_0(\theta_\Gamma \otimes I_\gamma)| = \beta_\Gamma$

and the restriction of ψ_0 to $N_0(\theta_\Gamma \otimes I_\gamma)$ is irreducible by the remark of (3I). Thus by (4.1) $|N(\theta) : N_0(\theta)| = \beta_\Gamma$. Now the restriction of ψ to

$$H = (N_0(\theta_\Gamma \otimes I_\gamma)/R_{m_\Gamma, \alpha_\Gamma, \gamma}) \times \mathrm{GL}(c_1, r) \times \mathrm{GL}(c_2, r) \times \cdots \times \mathrm{GL}(c_l, r)$$

is irreducible. Since $N_0(\theta)/R \geq H$, $\psi|_{N_0(\theta)/R}$ is irreducible, and so $\psi|_{N_0(\theta)}$ is irreducible.

Given $\Gamma \in \mathscr{F}'$ and integer $w_\Gamma \geq 1$, let $G = \mathrm{U}(V)$ or $I(V)$ and $G_0 = G$ or $I_0(V)$, where in the former case V is a unitary space of dimension $w_\Gamma e_\Gamma d_\Gamma$ over $\mathbb{F}_{q^2}$, in the latter case V is a symplectic or orthogonal space over $\mathbb{F}_q$ such that $\dim V = w_\Gamma \beta_\Gamma e_\Gamma d_\Gamma$ and if V is orthogonal, then $\eta(V) = \varepsilon^{w_\Gamma}$ or $\varepsilon_\Gamma^{w_\Gamma e_\Gamma}$ according as $\Gamma \in \mathscr{F}_0$ or $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$. Thus if V is unitary, then $s = w_\Gamma e_\Gamma \Gamma$ is a semisimple element of G and $C_G(s) \simeq \mathrm{GL}(w_\Gamma e_\Gamma, \varepsilon_\Gamma q^{\delta_\Gamma})$, so that G has a block B labeled by $(s, -)$ and a defect group D of B acts fixed-point freely on V since we may suppose D is a Sylow r-subgroup of $C_G(s)$. In the remaining case, a semisimple element s^* in G_0 exists such that $m_\Gamma(s^*) = \beta_\Gamma e_\Gamma w_\Gamma$ and $\eta_\Gamma(s^*) = \eta(V)$, so that a primary element of a Sylow r-subgroup of $C_G(s^*)$ exists and by the remark of (3E) a dual s of s^* exists in G_0^* which is uniquely determined in $I(V^*)$ up to conjugacy, where V^* is the underlying space of G_0^*. Moreover, by (3K) s uniquely determines a block B of G which covers a block in $\mathscr{E}_r(G_0, (s))$ and whose defect group acts fixed-point freely on V.

For each $\Gamma \in \mathscr{F}'$ and integer $d \geq 0$, let $\mathscr{C}_{\Gamma, d} = \{\varphi_{\Gamma, d, i, j} : 1 \leq i \leq \beta_\Gamma e_\Gamma,\ 1 \leq j \leq r^d\}$ be the set of characters associated with basic subgroups of $G = \mathrm{U}(V_{\Gamma, d})$ or $I(V_{\Gamma, d})$ in (4A).

(4B). *With the preceding notation, let B be a block of G with defect group D and root block $\mathbf{b}$ such that $[V, D] = V$ and $\mathbf{b}^{G_0} \subseteq \mathscr{E}_r(G_0, (s))$. Then the number of B-weights is the number f_Γ of assignments*

$$\coprod_{d \geq 0} \mathscr{C}_{\Gamma, d} \to \{r\text{-cores}\}, \qquad \varphi_{\Gamma, d, i, j} \mapsto \kappa_{\Gamma, d, i, j},$$

such that

$$\sum_{d \geq 0} r^d \sum_{i=1}^{\beta_\Gamma e_\Gamma} \sum_{j=1}^{r^d} |\kappa_{\Gamma, d, i, j}| = w_\Gamma.$$

Proof. Let (R, φ) be a B-weight of G, $C = C_G(R)$, and $N = N_G(R)$. Then there exists a block b of CR with defect group R such that $b^G = B$ and $\varphi \in b^N$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$. Let z be a primary element of D defined by the remark of (2D). Then $z \in Z(D)$ and $[V, z] = V$, so that $C_V(R) = 0$. Thus in the decomposition (2B) or (2D) of R, we may suppose

$$R = R_1^{d_1} \times R_2^{d_2} \times \cdots \times R_u^{d_u},$$

where R_i's are distinct nontrivial basic subgroups and R_i appears d_i times as a component of R. Let V_i be the underlying space of R_i, $G_i = \mathrm{U}(V_i)$ or $I(V_i)$ according as V_i is or is not a unitary space, $C_i = C_{G_i}(R_i)$, and $N_i = N_{G_i}(R_i)$. Then $C = C_1^{d_1} \times C_2^{d_2} \times \cdots \times C_u^{d_u}$. Let θ be the canonical character of b, so that we may suppose $\theta = \prod_{i=1}^u \theta_i^{d_i}$, where θ_i is an irreducible character of $C_i R_i$ trivial on R_i. Let z_i be the restriction of z on V_i and $K_i = C_{G_i}(z_i)$ for all i. Then K_j and $\prod_{i=1}^u K_i^{d_i}$ are a regular subgroup of $I_0(V_j)$ and G_0, so that $\prod_{i=1}^{u}(K_i^*)^{d_i}$ is embedded as a regular subgroup of G_0^*. If $(R_i, s_i, -)$ is a label of the Brauer pair (R_i, θ_i), then $s_i \in K_i^*$, $\prod_{i=1}^{u} s_i^{d_i} \in \prod_{i=1}^{u}(K_i^*)^{d_i}$, and so $(R, \prod_{i=1}^{u} s_i^{d_i}, -)$ is a label of the Brauer pair (R, b). Thus s_i and x_Γ are conjugate in $I_0(V_i)^*$, (R_i, θ_i) has type Γ, and $R_i = R_{m_\Gamma, \alpha_\Gamma, \gamma_i, \mathbf{c}_i}$ for some γ_i and $\mathbf{c}_i$. It is clear that

$$N(\theta) = \prod_{i=1}^{u} N(\theta_i) \wr \mathbf{S}(d_i),$$

where $N(\theta_i)$ is the stabilizer of θ_i in N_i. In particular, if $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$, then $\psi = \prod_{i=1}^{u} \psi_i$, where ψ_i is an irreducible character of $N(\theta_i) \wr \mathbf{S}(d_i)$ covering $\theta_i^{d_i}$. Moreover, ψ_i has defect 0 as a character of

$$N(\theta_i) \wr \mathbf{S}(d_i)/R_i^{d_i} \simeq (N(\theta_i)/R_i) \wr \mathbf{S}(d_i).$$

Let $\operatorname{Irr}^0(N(\theta_i), \theta_i) = \{\varphi_{i,j}\colon 1 \le j \le \beta_\Gamma e_\Gamma (r-1)^{l(R_i)}\}$. As shown in the proof of [3, (2C)], the irreducible characters of defect 0 of $(N(\theta_i)/R_i) \wr \mathbf{S}(d_i)$ covering $\theta_i^{d_i}$ are in bijection with assignments $\varphi_{i,j} \mapsto \kappa_{i,j}$ of characters to r-cores such that $\sum_{j \ge 1} |\kappa_{i,j}| = d_i$. Thus the irreducible characters of $\operatorname{Irr}^0(N(\theta), \theta)$ are in bijection with assignments $\varphi_{i,j} \mapsto \kappa_{i,j}$ of characters to r-cores such that

$$\sum_{i=1}^{u} (\deg R_i) \sum_{j \ge 1} |\kappa_{i,j}| = \beta_\Gamma e_\Gamma d_\Gamma w_\Gamma .$$

For fixed $d \ge 0$, the number of irreducible characters associated with basic groups of degree $\beta_\Gamma e_\Gamma d_\Gamma r^d$ is $\beta_\Gamma e_\Gamma r^d$. Let $\mathscr{C}_{\Gamma, d} = \{\varphi_{\Gamma, d, i, j}\colon 1 \le i \le \beta_\Gamma e_\Gamma,\ 1 \le j \le r^d\}$ be the set of these characters. Then the number of B-weights is the number of assignments

$$\coprod_{d \ge 0} \mathscr{C}_{\Gamma, d} \to \{r\text{-cores}\}, \qquad \varphi_{\Gamma, d, i, j} \mapsto \kappa_{\Gamma, d, i, j},$$

such that

$$\sum_{d \ge 0} \beta_\Gamma e_\Gamma r^d d_\Gamma \sum_{i=1}^{\beta_\Gamma e_\Gamma} \sum_{j=1}^{r^d} |\kappa_{\Gamma, d, i, j}| = \beta_\Gamma e_\Gamma d_\Gamma w_\Gamma .$$

This induces the required condition of (4B).

(4C). *With the preceding notation, let $G = \mathrm{O}(V)$ be an orthogonal group, $G_0 = \mathrm{SO}(V)$, and R a radical subgroup of G such that $[V, R] = V$. Let (R, b) a Brauer pair of G_0 labeled by $(R, s, -)$ and θ the canonical character of b. Then $|N(\theta) : N_0(\theta)| = \beta_\Gamma$ and the restriction $\psi|_{N_0(\theta)}$ of each $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$ to $N_0(\theta)$ is irreducible, where $N_0(\theta) = N(\theta) \cap G_0$.*

Proof. In the notation above $R = R_1^{d_1} \times R_2^{d_2} \times \cdots \times R_u^{d_u}$, V_i is the underlying space of R_i, $C = C_G(R) = \prod_{i=1}^{u} C_i^{d_i}$, and $\theta = \prod_{i=1}^{u} \theta_i^{d_i}$, where θ_i is an irreducible character of $C_i = C_{\mathrm{O}(V_i)}(R_i)$ for $i \ge 1$. Each (R_i, θ_i) has type Γ. Let $N(\theta_i)$ and $N_0(\theta_i)$ be the stabilizers of θ_i in $N_{\mathrm{O}(V_i)}(R_i)$ and $N_{\mathrm{SO}(V_i)}(R_i)$ respectively. By the remark of (4A), $|N(\theta_i) : N_0(\theta_i)| = \beta_\Gamma$ and so $|N(\theta) : N_0(\theta)| = \beta_\Gamma$ since $N(\theta) = \prod_{i=1}^{u} N(\theta_i) \wr \mathbf{S}(d_i)$. If $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$, then $\psi = \prod_{i=1}^{u} \psi_i$, where ψ_i is an irreducible character of $N(\theta_i) \wr \mathbf{S}(d_i)$ covering $\theta_i^{d_i}$. Moreover,

ψ_i has defect 0 as a character of $N(\theta_i) \wr \mathbf{S}(d_i)/R_i^{d_i} \simeq (N(\theta_i)/R_i) \wr \mathbf{S}(d_i)$. Let $N_0(\theta_i^{d_i})$ be the subgroup of $N(\theta_i) \wr \mathbf{S}(d_i)$ of determinant 1. It then suffices to show that the restriction of ψ_i to $N_0(\theta_i^{d_i})$ is irreducible. Thus we may suppose $u = 1$ and $d = d_1$, so that $\theta = \theta_1^d$ and $N(\theta) = N(\theta_1) \wr \mathbf{S}(d)$. Since $|N(\theta) : N_0(\theta)| \leq 2$, $\psi|_{N_0(\theta)}$ is irreducible if and only if $N(\theta)$ stabilizes an irreducible constituent of $\psi|_{N_0(\theta)}$.

Let $T = N(\theta_1)$, $H = N(\theta) = T \wr \mathbf{S}(d)$, $X = T^d$ the base subgroup of H, $H_0 = N_0(\theta)$, and X_0 the subgroup of X of determinant 1. Then $H = X \rtimes \mathbf{S}(d)$ and $H_0 = X_0 \rtimes \mathbf{S}(d)$. We may suppose $|H : H_0| = 2$ and hence $|T : T_0| = 2$, where $T_0 = T \cap I_0(V_1)$. Morever, (R_1, θ_1) has type Γ and the restriction of each character in $\operatorname{Irr}^0(T, \theta_1)$ to T_0 is irreducible by the remark of (4A). As shown in the proof of [3, (2B)] (cf. also [17, 5.20]), the irreducible characters of H can be obtained as follows: Let $\operatorname{Irr} T = \{\xi^1, \xi^2, \dots, \xi^t\}$ be the complete set of irreducible characters of T, and ξ an irreducible character of X. Then $\mathbf{m} = (m_1, m_2, \dots, m_t)$ is called the *type* of ξ if m_i is the multiplicity of ξ^i as a factor of ξ. The stabilizer of ξ in H is $X\mathbf{S}_\mathbf{m}$, and ξ can be extended to an irreducible character $\tilde{\xi}$ of $X\mathbf{S}_\mathbf{m}$ (see [17, 5.13]), where $\mathbf{S}_\mathbf{m}$ is the Young subgroup of $\mathbf{S}(d)$ of type $\mathbf{m}$. By Clifford theory, all irreducible characters of $X\mathbf{S}_\mathbf{m}$ covering ξ have form $\tilde{\xi}\zeta$ and $\operatorname{Ind}_{X\mathbf{S}_\mathbf{m}}^H(\tilde{\xi}\zeta)$ is irreducible, where ζ is an irreducible character of $X\mathbf{S}_\mathbf{m}$ trivial on X. Moreover, these characters $\{\operatorname{Ind}_{X\mathbf{S}_\mathbf{m}}^H(\tilde{\xi}\zeta)\}$ consist of a complete set of irreducible characters of H as ξ runs over the representatives of conjugacy H-classes of $\operatorname{Irr} X$, and, while ξ is fixed, ζ runs over irreducible characters of $\mathbf{S}_\mathbf{m}$, where $\mathbf{m}$ is the type of ξ (see [17, 5.20]). In particular, $\operatorname{Ind}_{X\mathbf{S}_\mathbf{m}}^H(\tilde{\xi}\zeta)$ has defect 0 as a character of H/R if and only if ζ has defect 0, and ξ has defect 0 as a character of X/R. If $\operatorname{Ind}_{X\mathbf{S}_\mathbf{m}}^H(\tilde{\xi}\zeta) \in \operatorname{Irr}^0(H, \theta)$, then we may suppose ξ covers θ.

Suppose $\xi \in \operatorname{Irr}^0(X, \theta)$. Then the restriction $\xi_0 = \xi|_{X_0}$ is irreducible since $\xi|_{T_0^d}$ is irreducible by the remark of (4A). Let K be the stabilizer of ξ_0 in H_0. Then $X_0\mathbf{S}_\mathbf{m} \leq K$, where $\mathbf{m}$ is the type of ξ. We claim $X_0\mathbf{S}_\mathbf{m} = K$. Indeed if there exists $x \in K \backslash X_0\mathbf{S}_\mathbf{m}$, then we may suppose $x \in \mathbf{S}(d) \backslash \mathbf{S}_\mathbf{m}$, $\xi^x \neq \xi$, and $\xi^x|_{X_0} = \xi_0$, since $H_0 = X_0\mathbf{S}(d)$ and the stabilizer of ξ is $X\mathbf{S}_\mathbf{m}$. In particular, $d > 1$. Thus $\xi_i \neq \xi_i'$ for some ith components ξ_i and ξ_i' of ξ and ξ^x respectively and so $\xi_i(h) \neq \xi_i'(h)$ for some $h \in T$. Since $\xi|_{X_0} = \xi^x|_{X_0}$, h has determinant -1. Let $w = \operatorname{diag}\{w_1, w_2, \dots, w_d\} \in X$ such that $w_i = h = w_j$ for some $j \neq i$, and $w_k = 1$ for $k \neq i, j$. Then $w \in X_0$ and so $\xi(w) = \xi^x(w)$. But the ith components of $\xi(w)$ and $\xi^x(w)$ are $\xi_i(h)$ and $\xi_i'(h)$ respectively. This is impossible and the claim holds.

Since $\tilde{\xi}$ is an extension of ξ to $X\mathbf{S}_\mathbf{m}$, it follows $\tilde{\xi}|_{X_0} = \xi_0$ and hence $\tilde{\xi}|_{X_0\mathbf{S}_\mathbf{m}} = \tilde{\xi}_0$ is an extension of ξ_0 to $X_0\mathbf{S}_\mathbf{m}$. By Clifford theory again, each irreducible character of $X_0\mathbf{S}_\mathbf{m}$ covering ξ_0 has the form $\tilde{\xi}_0\chi$, where χ is an irreducible character of $X_0\mathbf{S}_\mathbf{m}$ trivial on X_0, and each irreducible character of H_0 covering ξ_0 has the form $\operatorname{Ind}_{X_0\mathbf{S}_\mathbf{m}}^{H_0}(\tilde{\xi}_0\chi)$. Now for $\psi \in \operatorname{Irr}^0(H, \theta)$, $\psi = \operatorname{Ind}_{X\mathbf{S}_\mathbf{m}}^H(\tilde{\xi}\zeta)$ for some irreducible character ξ of X with defect 0 as a character of X/R, and $\xi|_{X_0} = \xi_0$ is irreducible. Thus there is an irreducible constituent ψ_0 of $\psi|_{H_0}$ covering ξ_0 and so $\psi_0 = \operatorname{Ind}_{X_0\mathbf{S}_\mathbf{m}}^{H_0}(\tilde{\xi}_0\chi)$. We claim that $\psi_0^\tau = \psi_0$ for any $\tau \in X$. Indeed this is true for $\tau \in X_0$ and we may suppose τ has determinant -1. Since $|X\mathbf{S_m} : X_0\mathbf{S_m}| \le 2$, τ normalizes $X_0\mathbf{S_m}$ and for $x, h \in H_0$, we have $h^{\tau^{-1}x} \in X_0\mathbf{S_m}$ if and only if $h^x \in X_0\mathbf{S_m}$ since $h^{\tau^{-1}x} = h^{x(x^{-1}\tau^{-1}x)}$ and $x^{-1}\tau^{-1}x \in X$. If $h^{\tau^{-1}x} \in X_0\mathbf{S_m}$, then $(\tilde{\xi}_0\chi)(h^{\tau^{-1}x}) = (\tilde{\xi}_0\chi)^{\tau'}(h^x)$, where $\tau' = x^{-1}\tau x \in X$. Since $\tilde{\xi}|_{X_0\mathbf{S_m}} = \tilde{\xi}_0$ is irreducible and χ is trivial on X_0, $\tilde{\xi}_0^g = \tilde{\xi}_0$ and $\chi^g = \chi$ for any $g \in X$. Therfore $(\tilde{\xi}_0\chi)^{\tau'}(h^x) = (\tilde{\xi}_0\chi)(h^x)$ and so $(\tilde{\xi}_0\chi)(h^{\tau^{-1}x}) = (\tilde{\xi}_0\chi)(h^x)$, for any $h, x \in H_0$. Thus $\psi_0^\tau = \psi_0$ and so $\psi|_{H_0} = \psi_0$ is irreducible. This proves (4C).

We now prove the main theorem of unitary groups.

(4D). *Let V be a unitary space over $\mathbb{F}_{q^2}$, $G = U(V)$, B be a block of G with label (s, κ), $\prod_\Gamma s(\Gamma)$ the primary decomposition of s, $\sum_\Gamma V(\Gamma)$ the corresponding orthogonal decomposition of V, and w_Γ the integer such that* $\dim V(\Gamma) - d_\Gamma|\kappa_\Gamma| = d_\Gamma e_\Gamma w_\Gamma$. *Then the following hold*:

(1) *The number of B-weights of G is $\prod_\Gamma f_\Gamma$, where f_Γ is given by* (4B). *In particular, f_Γ is the number of e_Γ-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{e_\Gamma} |\kappa_i| = w_\Gamma$.*

(2) *The number of B-weights of G is the number $l(B)$ of irreducible modular characters in B.*

Proof. Let R be a radical subgroup of G and $V = V_0 \perp V_+$, where $V_0 = C_V(R)$ and $V_+ = [V, R]$. Then $R = R_0 \times R_+$, where $R_0 = \langle 1_{V_0}\rangle$ and $R_+ \le \mathrm{U}(V_+)$. Let $C = C_G(R)$, $N = N_G(R)$, so that $C = C_0 \times C_+$, $N = N_0 \times N_+$, where $C_0 = N_0 = \mathrm{U}(V_0)$, $C_+ = C_{\mathrm{U}(V_+)}(R_+)$ and $N_+ = N_{\mathrm{U}(V_+)}(R_+)$. Suppose b is a block of CR with defect group R and $b^G = B$. Then $b = b_0 \times b_+$, where b_0 is a block of $C_0R_0 = \mathrm{U}(V_0)$ of defect 0, and b_+ is a block of C_+R_+ with defect group R_+. The canonical character θ of b decomposes as $\theta_0 \times \theta_+$, where θ_0 and θ_+ are the canonical characters of b_0 and b_+ respectively. Thus $N(\theta) = N_0 \times N(\theta_+)$, where $N(\theta_+)$ is the stabilizer of θ_+ in N_+.

Suppose $(R, I(\psi))$ is a B-weight of G, for some $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$. Clearly $\psi = \psi_0 \times \psi_+$ for character ψ_0 of N_0 and $\psi_+ \in \mathrm{Irr}^0(N(\theta_+), \theta_+)$. Since ψ_0 is a character of $N_0 = C_0$ covering θ_0, it follows that $\psi_0 = \theta_0$. The correspondence $(R, I(\psi)) \mapsto (R_+, I_+(\psi_+))$, where $\psi = \theta_0 \times \psi_+$ and $I_+(\psi_+) = \mathrm{Ind}_{N(\theta_+)}^{N_+}(\psi_+)$, is clearly a bijection from $\{(R, I(\psi)) \colon \psi \in \mathrm{Irr}^0(N(\theta), \theta)\}$ to $\{(R_+, I_+(\psi_+)) \colon \psi_+ \in \mathrm{Irr}^0(N(\theta_+), \theta_+)\}$.

By a theorem of Broué-Puig, [7, 3.2], we may suppose $s = s_0 \times s_+$ such that $s_0 \in C_0$, $s_+ \in C_+$, (s_0, κ) is the label of b_0, and $(s_+, -)$ is the label of $b_+^{\mathrm{U}(V_+)}$. In the correspondence above, $(R_+, I_+(\psi_+))$ is a $b_+^{\mathrm{U}(V_+)}$-weight. So the number of B-weights in G is the number of $b_+^{\mathrm{U}(V_+)}$-weights in $\mathrm{U}(V_+)$. Thus we may suppose $V = V_+$.

Let $R = \prod_{i=1}^t R_i$ and $V = \bigoplus_{i=1}^t V_i$ be the decompositions of (2B), and let $C = \prod_{i=1}^t C_i$ and $\theta = \prod_{i=1}^t \theta_i$, where $C_i = C_{\mathrm{U}(V_i)}(R_i)$ and θ_i is a character of C_i. Since the block b_i of C_iR_i containing θ_i has a defect group R_i, (R_i, θ_i) has type Γ for a unique $\Gamma \in \mathscr{F}'$ by (3B). Moreover, if $(R_i, t_i, -)$ is the label of (R_i, b_i), then $(R, \prod_i t_i, -)$ is the label of Brauer pair (R, b) of G, where b_i and b are regarded as blocks of C_i and C respectively. By [7, (3.2)] $(R, s, -)$ is also a label of (R, b), so that s and $\prod_i t_i$ are conjugate in G. Let $R(\Gamma) = \prod_i R_i$, $C(\Gamma) = \prod_i C_i$, $\theta(\Gamma) = \prod_\Gamma \theta_i$, and $t(\Gamma) = \prod_i t_i$, where i runs over all $1 \le i \le t$ such that (R_i, θ_i) is of type Γ. Then $R = \prod_\Gamma R(\Gamma)$,

$\theta = \prod_\Gamma \theta(\Gamma)$, $C = \prod_\Gamma C(\Gamma)$, and $\prod_\Gamma t(\Gamma)$ is a primary decomposition of s in G. We may suppose $s(\Gamma) = t(\Gamma)$, so that $N(\theta) = \prod_\Gamma N(\theta(\Gamma))$, where $N(\theta(\Gamma))$ is the stabilizer of $\theta(\Gamma)$ in $N_{\mathrm{U}(V(\Gamma))}(R(\Gamma))$.

Each $\psi = \prod_\Gamma \psi(\Gamma)$, for $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$ and $\psi(\Gamma) \in \mathrm{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma))$. Let $b(\Gamma)$ be a block of $C(\Gamma)$ containing $\theta(\Gamma)$, and $B(\Gamma) = b(\Gamma)^{\mathrm{U}(V(\Gamma))}$. Then $B(\Gamma)$ is labeled by $(s(\Gamma), -)$ and $(R(\Gamma), I(\psi(\Gamma)))$ is a $B(\Gamma)$-weight. Conversely, if $B(\Gamma)$ is a block of $\mathrm{U}(V(\Gamma))$ with label $(s(\Gamma), -)$ and $(R(\Gamma), \varphi(\Gamma))$ is a $B(\Gamma)$-weight, then there exists a block $b(\Gamma)$ of $C(\Gamma)R(\Gamma)$ with defect group $R(\Gamma)$ and the canonical character $\theta(\Gamma)$ such that $b(\Gamma)^{\mathrm{U}(V(\Gamma))} = B(\Gamma)$ and $\varphi(\Gamma) = I(\psi(\Gamma))$ for some $\psi(\Gamma) \in \mathrm{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma))$. Let $R = \prod_\Gamma R(\Gamma)$, $\theta = \prod_\Gamma \theta(\Gamma)$, $b = \prod_\Gamma b(\Gamma)$, and $\psi = \prod_\Gamma \psi(\Gamma)$. Then $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$, $b^G = B$, and $(R, I(\psi))$ is a B-weight. By (4B) the number of $B(\Gamma)$-weights of $\mathrm{U}(V(\Gamma))$ is f_Γ and so the number of B-weights of G is $\prod_\Gamma f_\Gamma$. By [3, (1A)] f_Γ is also the number of e_Γ-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{e_\Gamma} |\kappa_i| = w_\Gamma$. This last number is also the number of partitions with e_Γ-core κ_Γ and e_Γ-weight w_Γ. So $\prod_\Gamma f_\Gamma$ is the number $l(B)$ of irreducible modular characters in B by [11, (8A)]. This completes the proof.

(4E). *Let q be a power of an odd prime, V be a symplectic or even dimensional orthogonal space over $\mathbb{F}_q$, $G = I(V)$, $G_0 = I_0(V)$, B a block of G with defect group D and root block* $\mathbf{b}$ *such that $[V, D] = V$ and $\mathbf{b}^{G_0} \subseteq \mathscr{E}_r(G_0, (s))$ for some $s \in G_0^*$. Let s^* be a dual of s in G_0 and $m_\Gamma(s^*) = w_\Gamma \beta_\Gamma e_\Gamma$, where w_Γ is an integer and $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$. Then the number of B-weights is $\prod_\Gamma f_\Gamma$, where f_Γ is given by* (4B). *In particular, the number f_Γ is the numbler of $\beta_\Gamma e_\Gamma$-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{\beta_\Gamma e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{\beta_\Gamma e_\Gamma} |\kappa_i| = w_\Gamma$.*

Proof. Let (R, φ) be a B-weight of G, $C = C_G(R)$, and $N = N_G(R)$. Then there is a block b of CR with defect group R and the canonical character θ such that $b^G = B$ and $\varphi = I(\psi)$ for some $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$, so that $[V, R] = V$.

Let $R = \prod_i^t R_i$ and $V = \sum_{i=1}^t V_i$ be the decompositions of (2D), and let $C = \prod_{i=1}^t C_i$, and $\theta = \prod_{i=1}^t \theta_i$, where $C_i = C_{I(V_i)}(R_i)$ and θ_i is a character of $C_i R_i$ for all i. The block b_i of $C_i R_i$ containing θ_i has defect group R_i. We claim that there is a weight (R_i, χ_i) of $I(V_i)$ such that χ_i covers θ_i, namely there is an irreducible character χ_i of N_i/R_i which covers θ_i and whose defect is 0, where $N_i = N_{I(V_i)}(R_i)$. Thus by (3J)(a) (R_i, θ_i) has type Γ for some $\Gamma \in \mathscr{F}'$. To prove the claim we rewrite the decomposition of R as $\prod_{j=1}^u R_j^{d_j}$, where R_j's are distinct basic subgroups and R_j appears d_j-times as a component of R. Then

$$N = \prod_{j=1}^{u} N_j \wr \mathbf{S}(d_j).$$

Thus $\varphi = \prod_{j=1}^u \varphi_j$ and $(R_j^{d_j}, \varphi_j)$ is a weight of $I(U_j)$, where U_j is the underlying space of $R_j^{d_j}$. So we may suppose $u = 1$ and $d = d_1$. Thus $R = R_1^d$, $N = N_1 \wr \mathbf{S}(d)$, and φ is a character of defect 0 of $N/R \simeq (N_1/R_1) \wr \mathbf{S}(d)$. As shown in the proof of (4C), the restriction of φ to the base group $(N_1/R_1)^d$ of

N/R has a constituent $(\xi_1, \xi_2, \dots, \xi_d)$ covering θ and each ξ_i has defect 0 as character of N_1/R_1. Thus ξ_i covers θ_i and the claim holds.

Let $(R_i, t_i, -)$ be the label of Brauer pair (R_i, b_i). As shown in the proof of (4B), $(R, \prod_{i=1}^t t_i, -)$ is a label of (R, b) and $b^{G_0} \subseteq \mathscr{E}_r(G_0, (\prod_i t_i))$. If V^* is the underlying space of G_0^*, then s and $\prod_{i=1}^t t_i$ are conjugate in $I(V^*)$ by (3K).

Let $R(\Gamma) = \prod_i R_i$, $V(\Gamma) = \sum_i V_i$, $C(\Gamma) = \prod_i C_i$, $\theta(\Gamma) = \prod_i \theta_i$, and $t(\Gamma) = \prod_i t_i$, where i runs over $1 \le i \le t$ such that (R_i, θ_i) is of type Γ. Then $R = \prod_\Gamma R(\Gamma)$, $V = \sum_\Gamma V(\Gamma)$, $C = \prod_\Gamma C(\Gamma)$, $\theta = \prod_\Gamma \theta(\Gamma)$, and $\prod_\Gamma t(\Gamma)$ is conjugate with s in $I(V^*)$. It is clear that $N(\theta) = \prod_\Gamma N(\theta(\Gamma))$, where $N(\theta(\Gamma))$ is the stabilizer of $\theta(\Gamma)$ in $N_{I(V(\Gamma))}(R(\Gamma))$. A similar proof to the last paragraph of (4D) shows that the number of B-weights is $\prod_\Gamma f_\Gamma$ and by [3, (1A)] f_Γ is the number of $\beta_\Gamma e_\Gamma$-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{\beta_\Gamma e_\Gamma})$ of partitions κ_i such that $\sum_i |\kappa_i| = w_\Gamma$. This completes the proof.

Remark. With the assumption of (4E), let $G = \mathrm{O}(V)$, $G_0 = \mathrm{SO}(V)$, (R, φ) a B-weight of G, and θ an irreducible character of $C = C_G(R)$ covered by φ. Then $|N(\theta) \colon N_0(\theta)| = 1$ of 2 according as $m_{X\pm1}(s) = 0$ or $m_{X\pm1}(s) \ne 0$. Moreover, for each $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$, the restriction $\psi|_{N_0(\theta)}$ is irreducible, where $N_0(\theta) = N(\theta) \cap G_0$. Indeed in the notation above $R = \prod_\Gamma R(\Gamma)$, $V = \sum_\Gamma V(\Gamma)$, $\theta = \prod_\Gamma \theta(\Gamma)$, $N(\theta) = \prod_\Gamma N(\theta(\Gamma))$, and $s = \prod_\Gamma t(\Gamma)$. Thus $\psi = \prod_\Gamma \psi(\Gamma)$ for some $\psi(\Gamma) \in \mathrm{Irr}^0(N(\theta(\Gamma)), \theta(\Gamma))$. Since $[V, R] = V$, it follows that $[V(\Gamma), R(\Gamma)] = V(\Gamma)$. If $b(\Gamma)$ is the block of $C_{\mathrm{O}(V(\Gamma))}(R(\Gamma))R(\Gamma)$ containing $\theta(\Gamma)$, then the Brauer pair $(R(\Gamma), \theta(\Gamma))$ has label $(R(\Gamma), t(\Gamma), -)$. By (4C) $|N(\theta(\Gamma)) \colon N_0(\theta(\Gamma))| = \beta_\Gamma$ and $\psi(\Gamma)|_{N_0(\theta(\Gamma))}$ is irreducible, where $N_0(\theta(\Gamma)) = N(\theta(\Gamma)) \cap \mathrm{SO}(V(\Gamma))$. So $|N(\theta) : N_0(\theta)| = 1$ or 2 according as $m_{X\pm1}(s) = 0$ or $m_{X\pm1}(s) \ne 0$, and $\psi|_{N_0(\theta)}$ is irreducible.

(4F). *Let q be a power of an odd prime, $G = \mathrm{Sp}(2n, q) = \mathrm{Sp}(V)$, B a block of G contained in $\mathscr{E}_r(G, (s))$ for some semisimple r'-element s of $G^* = \mathrm{SO}(2n+1, q)$. Let D be a defect group of B, $V_0 = C_V(D)$, $V_+ = [V, D]$, so that $V = V_0 \perp V_+$, and let $s = s_0 \times s_+$ be the corresponding decomposition in G^*. Then $m_\Gamma(s) - m_\Gamma(s_0) = w_\Gamma \beta_\Gamma e_\Gamma$ for some $w_\Gamma \ge 0$, where $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$. The number of B-weights is $\prod_\Gamma f_\Gamma$, where f_Γ is given by* (4B). *In particular, f_Γ is the number of $\beta_\Gamma e_\Gamma$-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{\beta_\Gamma e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{\beta_\Gamma e_\Gamma} |\kappa_i| = w_\Gamma$.*

Proof. Let $(D, \mathbf{b})$ be a maximal Brauer pair of G containing $(1, B)$, and ϑ be the canonical character of $\mathbf{b}$. Then $D = D_0 \times D_+$, $\mathbf{b} = \mathbf{b}_0 \times \mathbf{b}_+$, and $\vartheta = \vartheta_0 \times \vartheta_+$, where $D_0 = \langle 1_{V_0} \rangle \le \mathrm{Sp}(V_0)$, $D_+ \le \mathrm{Sp}(V_+)$, $\mathbf{b}_0, \mathbf{b}_+$ are blocks of $\mathrm{Sp}(V_0)$ and $C_{\mathrm{Sp}(V_+)}(D_+)$ respectively, and $\vartheta_0 \in \mathbf{b}_0$, $\vartheta_+ \in \mathbf{b}_+$.

Let (R, φ) be a B-weight of G, $C = C_G(R)$, and $N = N_G(R)$. Then there is a block b of CR with defect group R and canonical character θ such that $b^G = B$ and $\varphi = I(\psi)$ for some $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$. We may suppose $Z(D) \le Z(R) \le R \le D$. Thus $C_V(R) = V_0$, $[V, R] = V_+$, so that $R = R_0 \times R_+$, $C = C_0 \times C_+$, $N = N_0 \times N_+$, where $R_0 = D_0$, $R_+ \le \mathrm{Sp}(V_+)$, $C_0 = N_0 = \mathrm{Sp}(V_0)$, $C_+ = C_{\mathrm{Sp}(V_+)}(R_+)$, and $N_+ = N_{\mathrm{Sp}(V_+)}(R_+)$. Let $b = b_0 \times b_+$ and $\theta = \theta_0 \times \theta_+$ be the corresponding decompositions. Then b_0 is a block of $C_0R_0 = \mathrm{Sp}(V_0)$ of defect 0, b_+ is a block of C_+R_+ with defect group R_+, and $\theta_0 \in b_0$, $\theta_+ \in b_+$. We claim $\theta_0 = \vartheta_0$. Indeed let $(D'_+, \mathbf{b}'_+)$ be a maximal Brauer pair of $\mathrm{Sp}(V_+)$ containing (R_+, b_+), $\mathbf{b}' = b_0 \times \mathbf{b}'_+$, and $D' = D_0 \times D'_+$. Then $(D', \mathbf{b}')$ is a maximal Brauer pair of $\mathrm{Sp}(V_0) \times \mathrm{Sp}(V_+)$ containing (R, b). If $N(D', \mathbf{b}')$ is the stabilizer of $(D', \mathbf{b}')$ in the normalizer $N_G(D')$ of D', then $(D', \mathbf{b}')$ is maximal in G if and only if $(D', \mathbf{b}')$ is maximal in $N(D', \mathbf{b}')$. Since $N_G(D') \leq \mathrm{Sp}(V_0) \times \mathrm{Sp}(V_+)$, $(D', \mathbf{b}')$ is maximal in $N(D', \mathbf{b}')$ and then maximal in G containing $(1, B)$. By the Brauer First Main Theorem, $(D, \mathbf{b})^g = (D', \mathbf{b}')$ for some $g \in G$, so that $(\vartheta_0 \times \vartheta_+)^g = \theta_0 \times \vartheta'_+$, where ϑ'_+ is the canonical charcter of $\mathbf{b}'_+$. Since $D = D_0 \times D_+$ and $D' = D_0 \times D'_+$, it follows $g \in \mathrm{Sp}(V_0) \times \mathrm{Sp}(V_+)$, and so $g = g_0 \times g_+$, for $g_0 \in \mathrm{Sp}(V_0)$ and $g_+ \in \mathrm{Sp}(V_+)$. Thus $\theta_0 = \vartheta_0$ and $\vartheta_+^{g_+} = \vartheta'_+$. Moreover, $b_+^{\mathrm{Sp}(V_+)} = \mathbf{b}_+^{\mathrm{Sp}(V_+)} = \mathbf{b}_+'^{\mathrm{Sp}(V_+)}$.

It is clear that $N(\theta) = N_0 \times N(\theta_+)$, where $N_0 = \mathrm{Sp}(V_0)$ and $N(\theta_+)$ is the stabilizer of θ_+ in N_+. If $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$, then $\psi = \psi_0 \times \psi_+$, where ψ_0 is an irreducible character of $N_0 = C_0$ covering θ_0, and $\psi_+ \in \mathrm{Irr}^0(N(\theta_+), \theta_+)$, so that $\psi_0 = \theta_0 = \vartheta_0$. The correspondence $(R, I(\psi)) \mapsto (R_+, I_+(\psi_+))$, where $\psi = \theta_0 \times \psi_+$ and $I_+(\psi_+) = \mathrm{Ind}_{N(\theta_+)}^{N_+}(\psi_+)$, is clearly a bijection form $\{(R, I(\psi)) \colon \psi \in \mathrm{Irr}^0(N(\theta), \theta)\}$ to $\{(R_+, I_+(\psi_+)) \colon \psi_+ \in \mathrm{Irr}^0(N(\theta_+), \theta_+)\}$. Since $(R_+, I_+(\psi_+))$ is a $\mathbf{b}_+^{\mathrm{Sp}(V_+)}$-weight, the number of B-weights in G is the number of $b_+^{\mathrm{Sp}(V_+)}$-weights in $\mathrm{Sp}(V_+)$. Thus (4E) implies (4F).

In the following, we consider special orthogonal groups. If $G = \mathrm{SO}(2n+1, q)$, then by Fong and Srinivasan, [12, (10B)], a block B of G is labeled by a pair (s, κ), where s is a semisimple r'-element in a dual group G^* of G, $\kappa = \prod_\Gamma \kappa_\Gamma$ is a product of symbols or partitions κ_Γ according as $\Gamma \in \mathscr{F}_0$ or $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ such that each κ_Γ is the e_Γ-core of either a symbol with rank $[\frac{1}{2} m_\Gamma(s)]$ and odd defect, or a partition of $m_\Gamma(s)$ according as $\Gamma \in \mathscr{F}_0$ or $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$. Moreover, by [12, (12A)], $B \subseteq \mathscr{E}_r(G, (s))$.

(4G). *Let q be a power of an odd prime, $G = \mathrm{SO}(V) = \mathrm{SO}(2n+1, q)$, B a block of G with label (s, κ), $\prod_\Gamma s(\Gamma)$ a primary decomposition of s in $G^* = \mathrm{Sp}(2n, q)$, and let w_Γ be an integer such that $m_\Gamma(s) = |\kappa_\Gamma| + e_\Gamma w_\Gamma$ if $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, and $m_\Gamma(s) = 2$ rank $\kappa_\Gamma + 2e_\Gamma w_\Gamma$ if $\Gamma \in \mathscr{F}_0$. Then the following hold:*

(1) *The number of B-weights of G is $\prod_\Gamma f_\Gamma$, where f_Γ is the number of $\beta_\Gamma e_\Gamma$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{\beta_\Gamma e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{\beta_\Gamma e_\Gamma} |\kappa_i| = w_\Gamma$, and $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$.*

(2) *The number of B-weights of G is $|B \cap \mathscr{E}(G, (s))|$.*

Proof. Let $\widetilde{G} = \mathrm{O}(V)$, so that $\widetilde{G} = \langle -1_V \rangle \times G$, and let $\widetilde{B} = 1 \times B$ be a block of $\widetilde{G}$, where 1 is the principal block of $\langle -1_V \rangle$. Let (R, φ) be a B-weight of G, $N = N_G(R)$, and $\widetilde{N} = N_{\widetilde{G}}(R)$, so that $\widetilde{N} = \langle -1_V \rangle \times N$. There exists a block b of N such that $\varphi \in b$ and $b^G = B$. Let $\tilde{b} = 1 \times b$ and $\tilde{\varphi} = 1_{\langle -1_V \rangle} \times \varphi$, where $1_{\langle -1_V \rangle}$ is the principal character of $\langle -1_V \rangle$. Thus $\tilde{\varphi} \in \tilde{b}$, $\tilde{b}^{\widetilde{G}} = \widetilde{B}$, and $(R, \tilde{\varphi})$ is a $\widetilde{B}$-weight of $\widetilde{G}$. The correspondence $(R, \varphi) \mapsto (R, \tilde{\varphi})$ is clearly a bijection from B-weights to $\widetilde{B}$-weights. Thus the number of B-weights in G is the number of $\widetilde{B}$-weights in $\widetilde{G}$.

Let $(D, \tilde{\mathbf{b}})$ be a maximal Brauer pair of $\widetilde{G}$ containing $(1, \widetilde{B})$, $\tilde{\vartheta}$ the canonical character of $\tilde{\mathbf{b}}$, $V_0 = C_V(D)$, $V_+ = [V, D]$. Then $V = V_0 \perp V_+$ and V_+ is an even dimensional orthogonal space since D is radical. In addition, let $\widetilde{G}_0 = \mathrm{O}(V_0)$, $G_0 = \mathrm{SO}(V_0)$, $\widetilde{G}_+ = \mathrm{O}(V_+)$, and $G_+ = \mathrm{SO}(V_+)$. Then

$D = D_0 \times D_+$, $\tilde{\mathbf{b}} = \mathbf{b}_0 \times \tilde{\mathbf{b}}_+$, $\tilde{\vartheta} = \tilde{\vartheta}_0 \times \tilde{\vartheta}_+$, where $D_0 = \langle 1_{V_0}\rangle \leq \widetilde{G}_0$, $D_+ \leq \widetilde{G}_+$, $\tilde{\mathbf{b}}_0$, $\tilde{\mathbf{b}}_+$ are blocks of $\widetilde{G}_0$, $C_{\widetilde{G}_+}(D_+)$ respectively, and $\tilde{\vartheta}_0 \in \tilde{\mathbf{b}}_0$, $\tilde{\vartheta}_+ \in \tilde{\mathbf{b}}_+$.

Now the proof of (4F) can be applied here with G replaced by $\widetilde{G}$, B by $\widetilde{B}$, ϑ by $\tilde{\vartheta}$, $\mathbf{b}$ by $\tilde{\mathbf{b}}$, and some obvious modifications. Thus the number of $\widetilde{B}$-weights in $\widetilde{G}$ is the number of $\tilde{\mathbf{b}}_+^{\widetilde{G}_+}$-weights in $\widetilde{G}_+$. Moreover, $\tilde{\mathbf{b}}_+$ is a root block of $\tilde{\mathbf{b}}_+^{\widetilde{G}_+}$ and $\tilde{\mathbf{b}}^{G_+} \subseteq \mathscr{E}_r(G_+, (s_+))$. Since $C_{\widetilde{G}}(D) = \langle -1_{V_0}\rangle \times C_G(D)$ and $\widetilde{B} = 1 \times B$, it follows that $\tilde{\mathbf{b}} = 1 \times \mathbf{b}$ for some block $\mathbf{b}$ of $C_G(D)$ and $\tilde{\mathbf{b}}_0 = 1 \times \mathbf{b}_0$, where 1 is the principal block of $\langle -1_{V_0}\rangle$ and $\mathbf{b}_0$ is a block of G_0. Thus $\tilde{\vartheta}_0 = 1_{\langle -1_{V_0}\rangle} \times \vartheta_0$ for $\vartheta_0 \in \mathbf{b}_0$. Since $C_{\widetilde{G}_+}(D_+) = C_{G_+}(D_+)$, $\tilde{\mathbf{b}}_+$ is a block of $C_{G_+}(D_+)$ and then $\mathbf{b}_0 \times \tilde{\mathbf{b}}_+$ is a root block of B. Here $\mathbf{b}_0 \times \tilde{\mathbf{b}}_+$ is regarded as a block of $C_G(D)D$. As shown in the proof of [12, (12A)], (s_0, κ) is the label of ϑ_0, so that $m_\Gamma(s) = |\kappa_\Gamma| + m_\Gamma(s_+)$ if $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$, and $m_\Gamma(s) = 2$ rank $\kappa + m_\Gamma(s_+)$ if $\Gamma \in \mathscr{F}_0$. Thus $m_\Gamma(s_+) = m_\Gamma(s_+^*) = w_\Gamma \beta_\Gamma e_\Gamma$, where s_+^* is a dual of s_+ in G_+. So (4G)(1) follows from (4E).

Finally, there exists a bijection between $\mathscr{E}(G, (s))$ and $\mathscr{E}(C_{G^*}(s)^*, (1))$. By [12, (12A)] and [19, Proposition 14] the number given by (1) is the number of the characters of $\mathscr{E}(G, (s)) \cap B$.

Remark. (1) Suppose $G = \mathrm{SO}(2n+1, q)$ and r is a good prime. Then by [13, 5.1] $l(B) = |B \cap \mathscr{E}(G, (s))|$, so that $l(B)$ is the number of B-weights.

(2) By a result of Fong and Olsson (unpublished), if $G = \mathrm{SO}(2n+1, q)$ and r is odd, then $l(B) = |B \cap \mathscr{E}(G, (s))|$ and this is the number of B-weights.

(4H). *Let q be a power of an odd prime, $G = \mathrm{SO}^{\pm}(2n, q) = \mathrm{SO}(V)$, B is a block of G with defect group D and root block $\mathbf{b}$ such that $B \subseteq \mathscr{E}_r(G, (s))$ for some semisimple r'-element s of $G^* = \mathrm{SO}^{\pm}(2n, q)$, and let $V_0 = C_V(D)$, $V_+ = [V, D]$, so that $V = V_0 \perp V_+$. Let $s = s_0 \times s_+$, $\vartheta = \vartheta_0 \times \vartheta_+$ be the corresponding decompositions, where ϑ is the canonical character of $\mathbf{b}$. If $m_\Gamma(s_+) = w_\Gamma \beta_\Gamma e_\Gamma$ for some $w_\Gamma \geq 0$, then denote f_Γ the number of $\beta_\Gamma e_\Gamma$-tuples $(\kappa_1, \kappa_2, \dots, \kappa_{\beta_\Gamma e_\Gamma})$ of partitions κ_i such that $\sum_{i=1}^{\beta_\Gamma e_\Gamma} |\kappa_i| = w_\Gamma$, where $\beta_\Gamma = 1$ or 2 according as $\Gamma \in \mathscr{F}_1 \cup \mathscr{F}_2$ or $\Gamma \in \mathscr{F}_0$. Then the following hold:*

(1) *If either $m_{X\pm 1}(s_+) = 0$ or $\vartheta_0^{\sigma_0} = \vartheta_0$ for some $\sigma_0 \in \mathrm{O}(V_0)$ of determinant -1, then the number of B-weights is $\prod_\Gamma f_\Gamma$.*

(2) *Suppose $m_{X\pm 1}(s_+) \neq 0$. If either $V_0 = 0$ or $\vartheta_0^{\sigma_0} \neq \vartheta_0$ for any $\sigma_0 \in \mathrm{O}(V_0)$ of determinant -1, then the number of B-weights is $\frac{1}{2}\prod_\Gamma f_\Gamma$.*

Proof. Let $\widetilde{G} = \mathrm{O}(V)$, $\widetilde{G}_0 = \mathrm{O}(V_0)$, $G_0 = \mathrm{SO}(V_0)$, $\widetilde{G}_+ = \mathrm{O}(V_+)$, $G_+ = \mathrm{SO}(V_+)$, and $D = D_0 \times D_+$, where $D_0 = \langle 1_{V_0}\rangle$ and $D_+ \leq G_+$. In addition, let $\mathbf{b}_+$ be a block of $C_{G_+}(D_+)D_+$ containing ϑ_+, and $\mathbf{b}_+^{G_+} \subseteq \mathscr{E}_r(G_+, (s'_+))$ for some semisimple r'-element s'_+ of G_+^*. Then $(D_+, s'_+, -)$ is a label of Brauer pair $(D_+, \mathbf{b}_+)$. But $(D_+, s_+, -)$ is also a label of $(D_+, \mathbf{b}_+)$, and so s_+, s'_+ are conjugate in G_+^*.

Let (R, φ) be a B-weight, $C = C_G(R)$, $\widetilde{C} = C_{\widetilde{G}}(R)$, $N = N_G(R)$, and $\widetilde{N} = N_{\widetilde{G}}(R)$. Then there exists a block b of CR with defect group R and canonical character θ such that $b^G = B$ and $\varphi = I(\psi)$ for some $\psi \in \mathrm{Irr}^0(N(\theta), \theta)$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$, so that $R = R_0 \times R_+$, $C = G_0 \times C_+$, $\widetilde{C} = \widetilde{G}_0 \times C_+$, $N = \langle \tau, G_0 \times N_+\rangle$, and $\widetilde{N} = \widetilde{G}_0 \times \widetilde{N}_+$, where $R_0 = D_0$,

$R_+ \leq G_+$, $C_+ = C_{G_+}(R_+)$, $N_+ = N_{G_+}(R_+)$, $\widetilde{N}_+ = N_{\widetilde{G}_+}(R_+)$, and $\tau = \tau_0 \times \tau_+$ with $\tau_0 \in \widetilde{G}_0$, $\tau_+ \in \widetilde{G}_+$ of determinants -1. Thus $\widetilde{N} = \langle \tau_0, N \rangle$, $\theta = \theta_0 \times \theta_+$, and $b = b_0 \times b_+$, where b_0 is a block of G_0 of defect 0, b_+ is a block of C_+R_+ with defect group R_+, $\theta_0 \in b_0$, and $\theta_+ \in b_+$.

Let $(D'_+, \mathbf{b}'_+)$ be a maximal Brauer pair of $\widetilde{G}_+$ containing (R_+, b_+), where b_+ is regarded as a block of C_+. Let $D' = D_0 \times D'_+$, $\mathbf{b}' = b_0 \times \mathbf{b}'_+$. A similar proof to that of (4F) shows that $(D', \mathbf{b}')$ is a maximal Brauer pair of G containing (R, b), where b is regarded as a block of C. So $(D, \mathbf{b})^g = (D', \mathbf{b}')$ for some $g \in G$ by the Brauer First Main Theorem. Thus $g = g_0 \times g_+$ for $g_0 \in \widetilde{G}_0$ and $g_+ \in \widetilde{G}_+$. If $\det g_0 = -1$, then we replace b by b^τ and θ_0 by $\theta_0^{\tau_0}$. We may suppose $g_0 \in G_0$ and $g_+ \in G_+$. Since $(\vartheta_0 \times \vartheta_+)^g = \theta_0 \times \vartheta'_+$, it follows that $\theta_0 = \vartheta_0$ and $\vartheta_+^{g_+} = \vartheta'_+$, where ϑ'_+ is the canonical character of $\mathbf{b}'_+$. It follows that $\mathbf{b}'^{G_+}_+ = \mathbf{b}^{G_+}_+$, so that $\mathbf{b}'^{\widetilde{G}_+}_+ = b^{\widetilde{G}_+}_+$ and we may suppose $(R_+, s_+, -)$ is a label of (R_+, b_+). Replacing R by $R_0 \times R_+^{g_+^{-1}}$ and b by $b_0 \times b_+^{g_+^{-1}}$, we may suppose $(R, b) \leq (D, \mathbf{b})$.

(1) Suppose $m_{X\pm 1}(s_+) = 0$. Set $\widetilde{B}_+ = \mathbf{b}_+^{\widetilde{G}_+}$, so that $\mathbf{b}_+$ is a root block of $\widetilde{B}_+$ and D_+ is a defect group of $\widetilde{B}_+$. We shall show that the number of B-weights in G is the number of $\widetilde{B}_+$-weights in $\widetilde{G}_+$.

Let $N(\theta_+)$ and $\widetilde{N}(\theta_+)$ be the stabilizers of θ_+ in N_+ and $\widetilde{N}_+$ respectively. By the remark of (4E) $N(\theta_+) = \widetilde{N}(\theta_+)$. Since $N(\theta) = G_0 \times N(\theta_+)$, it follows that $\psi = \theta_0 \times \psi_+$ for some $\psi_+ \in \operatorname{Irr}^0(N(\theta_+), \theta_+)$. Then $(R_+, I_+(\psi_+))$ is a $\widetilde{B}_+$-weight of $\widetilde{G}_+$, where $I_+(\psi_+) = \operatorname{Ind}_{\widetilde{N}(\theta_+)}^{\widetilde{N}_+}(\psi_+)$. Conversely, suppose (R_+, φ_+) is a $\widetilde{B}_+$-weight, where R_+ is a radical subgroup of $\widetilde{G}_+$. Then $[V_+, R_+] = V_+$ and there exists a block of C_+R_+ with defect group R_+ and canonical character θ_+ such that $\varphi_+ = I_+(\psi_+)$ for some $\psi_+ \in \operatorname{Irr}^0(\widetilde{N}(\theta_+), \theta_+)$ and $b_+^{\widetilde{G}_+} = \widetilde{B}_+$, where C_+, $\widetilde{N}_+$ are given before, $\widetilde{N}(\theta_+)$ is the stabilizer of θ_+ in $\widetilde{N}_+$, and I_+ is defined as before. By the remark of (4E) $\widetilde{N}(\theta_+) \leq G_+$. Let $\theta = \vartheta_+ \times \theta_+$, $R = D_0 \times R_+$, $\psi = \vartheta_0 \times \psi_+$, b a block of $C_G(R)$ containing θ, and $N(\theta)$ the stabilizer of θ in $N = N_G(R)$. Then $N(\theta) = G_0 \times \widetilde{N}(\theta_+)$ and $\psi \in \operatorname{Irr}^0(N(\theta), \theta)$. We may suppose $(R_+, b_+) \leq (D_+, \mathbf{b}_+)$, so that $(R, b) \leq (D, \mathbf{b})$. Thus $b^G = B$ and $(R, I(\psi))$ is a B-weight. The correspondence $(R, I(\psi)) \mapsto (R_+, I_+(\psi_+))$, where $R = D_0 \times R_+$ and $\psi = \vartheta_0 \times \psi_+$ is clearly a bijection from $\{(R, I(\psi)) \colon \psi \in \operatorname{Irr}^0(N(\theta), \theta)\}$ to $\{(R_+, I_+(\psi_+)) \colon \psi_+ \in \operatorname{Irr}^0(\widetilde{N}(\theta_+), \theta_+)\}$. So the number of B-weights is the number of $\widetilde{B}_+$-weights, and it is $\prod_\Gamma f_\Gamma$ by (4E).

Suppose $\vartheta_0^{\sigma_0} = \vartheta_0$ for some $\sigma_0 \in \widetilde{G}_0$ of determinant -1. Then there are two irreducible characters ϑ'_0 and ϑ''_0 of $\widetilde{G}_0$ covering ϑ_0. Let $\vartheta' = \vartheta'_0 \times \vartheta_+$, $\vartheta'' = \vartheta''_0 \times \vartheta_+$, and $\mathbf{b}'$, $\mathbf{b}''$ be the blocks of $C_{\widetilde{G}}(D)$ containing ϑ', ϑ'' respectively. Then ϑ', ϑ'' are not conjugate in $N_{\widetilde{G}}(D) = \widetilde{G}_0 \times N_{\widetilde{G}_+}(D_+)$, so $\mathbf{b}'^{\widetilde{G}}$ and $\mathbf{b}''^{\widetilde{G}}$ are two blocks of $\widetilde{G}$. We shall show that the number of $\mathbf{b}'^{\widetilde{G}}$-weights is the number of B-weights.

Suppose (R, φ) is a B-weight. In the notation above, $N = \langle \tau, G_0 \times N_+ \rangle$

and $\widetilde{N} = \langle \tau_0, G_0 \times \widetilde{N}_+ \rangle$, where $\tau = \tau_0 \times \tau_+$ with $\tau_0 \in \widetilde{G}_0$, $\tau_+ \in \widetilde{G}_+$ of determinants -1. Moreover, we may suppose $(R, b) \le (D, \mathbf{b})$ and $\theta_0 = \vartheta_0$. Let $\tilde{\theta} = \vartheta'_0 \times \theta_+$ and $\tilde{b}$ the block of $\widetilde{C}$ containing $\tilde{\theta}$. Then $(R, \tilde{b}) \le (D, \mathbf{b}')$ and $\tilde{b}^{\widetilde{G}} = \mathbf{b}'^{\widetilde{G}}$. Conversely, if $(R, \tilde{\varphi})$ is a weight of $\mathbf{b}'^{\widetilde{G}}$, then there exists a block $\tilde{b}$ of $\widetilde{C}R$ with defect group R and canonical character $\tilde{\theta}$ such that $\tilde{b}^{\widetilde{G}} = \mathbf{b}'^{\widetilde{G}}$ and $\tilde{\varphi} \in \operatorname{Irr}(\widetilde{N}, \tilde{\theta})$, where $\widetilde{C}$ is defined before. Then $\tilde{b} = \tilde{b}_0 \times b_+$ and $\tilde{\theta} = \tilde{\theta}_0 \times \theta_+$, where $\tilde{b}_0$ and b_+ are blocks of $\widetilde{G}_0$ and C_+ respectively and $\tilde{\theta}_0 \in \tilde{b}_0$ and $\theta_+ \in b_+$. As shown in the proof of (4F), we may suppose $\tilde{\theta}_0 = \vartheta'_0$ and $(R, \tilde{b}) \le (D, \mathbf{b}')$. Let $\theta = \vartheta_0 \times \theta_+$ and b the block of C containing θ. Then $(R, b) \le (D, \mathbf{b})$. In addition, each character $\varphi \in \operatorname{Irr}^0(N, \theta)$ or $\tilde{\varphi} \in \operatorname{Irr}^0(\widetilde{N}, \tilde{\theta})$ covers a character of $\operatorname{Irr}^0(G_0 \times N_+, \theta)$ and each character of $\operatorname{Irr}^0(G_0 \times N_+, \theta)$ decomposes as $\vartheta_0 \times \varphi_+$ for some $\varphi_+ \in \operatorname{Irr}^0(N_+, \theta_+)$. So it suffices to show that the number of $\mathbf{b}'^{\widetilde{G}}$-weights of the form $(R, \tilde{\varphi})$ with $\tilde{\varphi}$ covering $\vartheta_0 \times \varphi_+$ is the number of B-weights of the form (R, φ) with φ covering $\vartheta_0 \times \varphi_+$. It is equivalent to show that the number of irreducible characters in $\tilde{b}^{\widetilde{N}}$ covering $\vartheta_0 \times \varphi_+$ is the number of irreducible characters in b^N covering $\vartheta_0 \times \varphi_+$ since $(\widetilde{N} : N) = (N : G_0 \times N_+) = 2$.

If τ_+ stabilizes φ_+, then there are two irreducible characters φ'_+ and φ''_+ of $\widetilde{N}_+$ covering φ_+, so that there are four irreducible characters $\vartheta'_0 \times \varphi'_+$, $\vartheta'_0 \times \varphi''_+$, $\vartheta''_0 \times \varphi'_+$, and $\vartheta''_0 \times \varphi''_+$ of $\widetilde{N} = \widetilde{G}_0 \times \widetilde{N}_+$ covering $\vartheta_0 \times \varphi_+$. Moreover, exactly two of them $\vartheta'_0 \times \varphi'_+$ and $\vartheta'_0 \times \varphi''_+$ cover $\vartheta'_0 \times \varphi_+$ and both lie in $\tilde{b}^{\widetilde{N}}$ by [10, V 3.10 and 3.7]. Since $\tau = \tau_0 \times \tau_+$ stabilizes $\vartheta_0 \times \varphi_+$, there are two irreducible characters of N covering $\vartheta_0 \times \varphi_+$ and lying in b^N. It follows that both $\tilde{b}^{\widetilde{N}}$ and b^N have two irreducible characters covering $\vartheta_0 \times \varphi_+$, so that the number of $\mathbf{b}'^{\widetilde{G}}$-weights is the number of B-weights.

If τ_+ does not stabilize φ_+, then there are two irreducible characters $\vartheta'_0 \times (\varphi_+ + \varphi_+^{\tau_+})$ and $\vartheta''_0 \times (\varphi_+ + \varphi_+^{\tau_+})$ of $\widetilde{N}$ covering $\vartheta_0 \times \varphi_+$ and only the first lies in $\tilde{b}^{\widetilde{N}}$. Since $(\vartheta_0 \times \varphi_+)^\tau \ne \vartheta_0 \times \varphi_+$, N has only one irreducible character covering $\vartheta_0 \times \varphi_+$ and lying in b^N. So both $\tilde{b}^{\widetilde{N}}$ and b^N has one irreducible character covering $\vartheta_0 \times \varphi_+$. Thus the number of $\mathbf{b}'^{\widetilde{G}}$-weights is the number of B-weights.

A similar proof to that of (4F) can be applied here with G replaced by $\widetilde{G}$, B by $\mathbf{b}'^{\widetilde{G}}$, $\mathbf{b}$ by $\mathbf{b}'$, ϑ by ϑ', and some obvious modifications, so that the number of $\mathbf{b}'^{\widetilde{G}}$-weights is the number of $\mathbf{b}_+^{\widetilde{G}_+}$-weights. By (4E) the number of $\mathbf{b}_+^{\widetilde{G}_+}$-weights is $\prod_\Gamma f_\Gamma$ and this is the number of B-weights. This completes the proof of (1).

(2) Suppose $m_{X\pm 1}(s_+) \ne 0$ and (R, φ) is a B-weight. In the notation above, suppose $\widetilde{N}(\theta)$ and $N(\theta)$ are the stabilizers of θ in $\widetilde{N}$ and N respectively.

If $V_0 = 0$, then $(\widetilde{N}(\theta) \colon N(\theta)) = 2$ and $|\operatorname{Irr}^0(\widetilde{N}(\theta), \theta)| = 2|\operatorname{Irr}^0(N(\theta), \theta)|$ by the remark of (4E). So the number of B-weights is $\frac{1}{2} \prod_\Gamma f_\Gamma$ by (4E).

Suppose $V_0 \ne 0$ and $\vartheta_0^{\tau_0} \ne \vartheta_0$ for some $\tau_0 \in \widetilde{G}_0$ of determinant -1. By the proof above, we may suppose $\theta = \vartheta_0 \times \theta_+$ for some character θ_+ of C_+ and $(R, b) \le (D, \mathbf{b})$. Let $\widetilde{N}(\theta_+)$ and $N(\theta_+)$ be the stabilizers of θ_+ in $\widetilde{N}_+$ and N_+ respectively. Then $\widetilde{N}(\theta) = G_0 \times \widetilde{N}(\theta_+)$ and $N(\theta) = G_0 \times N(\theta_+)$, so that by the remark of (4E), $|\mathrm{Irr}^0(\widetilde{N}(\theta_+), \theta_+)| = 2|\mathrm{Irr}^0(N(\theta_+), \theta_+)|$. Thus $|\mathrm{Irr}^0(\widetilde{N}(\theta), \theta)| = 2|\mathrm{Irr}^0(N(\theta), \theta)|$ since each character $\tilde{\psi}$ of $\mathrm{Irr}^0(\widetilde{N}(\theta), \theta)$ and each ψ of $\mathrm{Irr}^0(N(\theta), \theta)$ decomposes as $\tilde{\psi} = \vartheta_0 \times \tilde{\psi}_+$ and $\psi = \vartheta_0 \times \psi_+$ for some $\tilde{\psi}_+ \in \mathrm{Irr}^0(\widetilde{N}(\theta_+), \theta_+)$ and $\psi_+ \in \mathrm{Irr}^0(N(\theta_+), \theta_+)$. Let $\mathbf{b}'$ be the block of $C_{\widetilde{G}}(D)D$ containing $\vartheta' = (\vartheta_0 + \vartheta_0^{\tau_0}) \times \vartheta_+$ and $\tilde{b}$ the block of $\widetilde{C}$ containing $\tilde{\theta} = (\vartheta_0 + \vartheta_0^{\tau_0}) \times \theta_+$. Since $(R, b) \le (D, \mathbf{b})$ in G, it follows that $(R, \tilde{b}) \le (D, \mathbf{b}')$ in $\widetilde{G}$, so that $\tilde{b}^{\widetilde{G}} = \mathbf{b}'^{\widetilde{G}}$. Thus the number of B-weights is half of the number of $\mathbf{b}'^{\widetilde{G}}$-weights. A similar proof to that of (4F) can be applied here with G replaced by $\widetilde{G}$, B by $\mathbf{b}'^{\widetilde{G}}$, $\mathbf{b}$ by $\mathbf{b}'$, ϑ by ϑ', and some obvious modifications, so that the number of $\mathbf{b}'^{\widetilde{G}_+}$-weights is the number of $\mathbf{b}_+^{\widetilde{G}}$-weights. By (4E) the number of $\mathbf{b}_+^{\widetilde{G}_+}$-weights is $\prod_\Gamma f_\Gamma$ and so the number of B-weights is $\frac{1}{2}\prod_\Gamma f_\Gamma$. This completes the proof.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680

Current address: Department of Mathematics, University of Auckland, Auckland, New Zealand

E-mail address: an@mat.aukuni.ac.nz